A characterization of robust SPR synthesis for systems with l_p parametric uncertainty

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Outline

- Robust SPR with l_p uncertainty: problem statement
- Robust SPR vs. l_p parametric stability margin
- Frequency domain characterization
- Problem solution for the ellipsoidal (l_2) case
- Problem solution for the polyhedral (l_∞ and l_1) case
- Conclusion

Motivation and problem statement

- Robust absolute stability of nonlinear feedback Lur'e systems [Siljak 1969]
- Convergence of recursive identification algorithms [Ljung 1977] and adaptive schemes

Robust SPR problem. Given an uncertain set $\mathcal{P}$ of polynomials, find a polynomial or rational filter $F(s)$ such that

$$\frac{P(s)}{F(s)} \text{ is SPR } \forall P(s) \in \mathcal{P}$$

[Dasgupta et al. 1987], [Anderson et al. 1990]

l_p robust SPR problem

- l_p set of polynomials of degree m

$$\mathcal{P}_\rho^p := \left\{ P(s) = P_0(s) + \sum_{i=1}^{n} q_i P_i(s) : \|q\|_p \leq \rho \right\}$$

$$P_0(s) \in \mathcal{H}; \qquad \partial P_0 = m, \quad \partial P_i < m \quad \forall i = 1, \ldots, n;$$
$$q = (q_1 \ldots q_n)' \in \mathbb{R}^n; \qquad \rho > 0$$

- Parametric stability margin of the set $\mathcal{P}_\rho^p$

$$\rho^* = \sup_{\mathcal{P}_\rho^p \subset \mathcal{H}} \rho$$

i.e., the maximum l_p norm of the uncertain parameters such that $\mathcal{P}_\rho^p$ contains only Hurwitz polynomials.

l_p robust SPR problem

Theorem. [Anderson et al. 1990].
Given the set $\mathcal{P}_\rho^p$, suppose $\rho < \rho^*$. Then, there exist an integer M and a Hurwitz polynomial $R(s)$ of degree $m + M$ such that the filter

$$F(s) = \frac{R(s)}{(s+1)^M}$$

solves the RSPR problem.

- $F(s)$ can be approximated via a series expansion
  - No closed form for $F(s)$
  - No a-priori knowledge on the degree M of the filter $F(s)$

Characterization of l_p RSPR problem solutions

$$G(s) := \left(-\frac{P_1(s)}{P_0(s)}, \ldots, -\frac{P_n(s)}{P_0(s)}\right)'$$

$$R(\omega) := \mathrm{Re}[G(j\omega)], \quad I(\omega) := \mathrm{Im}[G(j\omega)].$$

Theorem. All rational filters solving the l_p RSPR are given by

$$F(s) = \frac{P_0(s)}{\Phi(s)}$$

where $\Phi(s)$ is a rational function such that

1. $\Phi(s)$ is SPR;
2. $\|R(\omega) - \gamma_\Phi(\omega) I(\omega)\|_p^d < \frac{1}{\rho} \quad \forall \omega \geq 0$

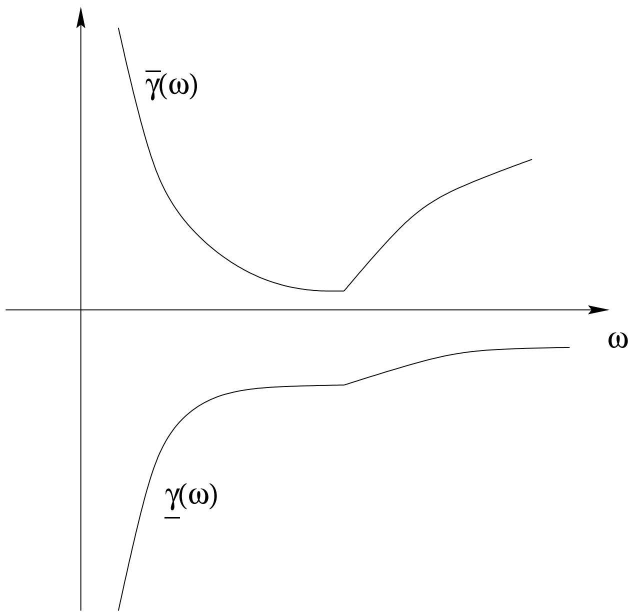
being

$$\gamma_\Phi(\omega) := \mathrm{Im}[\Phi(j\omega)]/\mathrm{Re}[\Phi(j\omega)]; \qquad \|\cdot\|_p^d: \text{ dual } p\text{-norm}$$

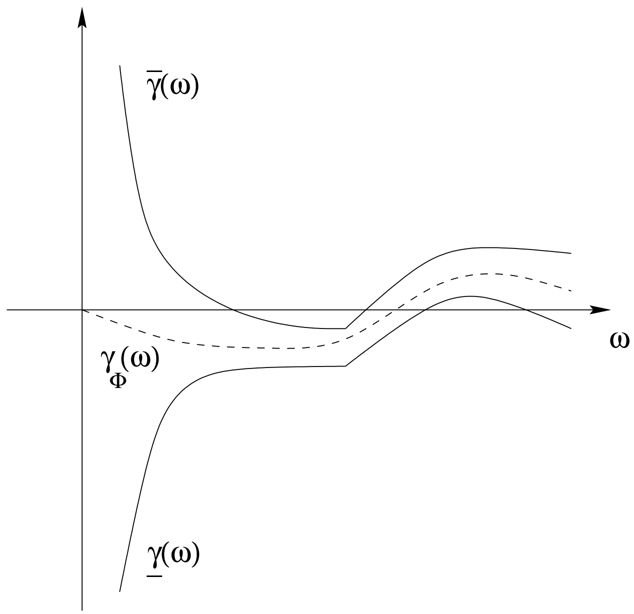
Characterization of l_p RSPR problem solutions

- Phase condition

$$\|R(\omega) - \gamma_\Phi(\omega) I(\omega)\|_p^d < \frac{1}{\rho} \quad \Leftrightarrow \quad \underline{\gamma}(\omega) < \gamma_\Phi(\omega) < \overline{\gamma}(\omega)$$



$\rho = \rho_1$



$\rho = \rho_2 < \rho_1$

l_p RSPR problem solution

- Key. Characterize the set

$$\Gamma_\rho^p = \left\{ \gamma(\omega) \;:\; \|R(\omega) - \gamma(\omega) I(\omega)\|_p^d < \frac{1}{\rho} \quad \forall \omega \geq 0 \right\}$$

- A solution exists if and only if a SPR function $\Phi(s)$ can be synthesized such that

$$\gamma_\Phi(\omega) \in \Gamma_\rho^p$$

- Idea. Find a SPR function $\Phi(s)$ such that

$$\gamma_\Phi(\omega) \approx \gamma^*(\omega) = \arg\min_\gamma \|R(\omega) - \gamma I(\omega)\|_p^d$$

- $\gamma^*(\omega)$ is independent of ρ

The ellipsoidal (l_2) case

- If $\rho < \rho^*$, then a positive real (PR) function $\Phi^*(s)$ exists such that

$$\gamma_{\Phi^*}(\omega) = \gamma^*(\omega)$$

- $\Phi^*(s)$ can be explicitly computed in closed form via the solution of a polynomial factorization problem
- The sought function $\Phi(s)$ can be computed by performing a suitable perturbation on $\Phi^*(s)$

The l_2 case

Simplifying assumption (can be relaxed). The set $\mathcal{P}_\rho^2$ is such that

$$I(\omega) \neq 0 \quad \forall \omega > 0.$$

- The polynomial

$$\Pi(s) = \sum_{i=1}^{n} P_0(s)P_i(-s)\left[P_0(-s)P_i(s)\right]_{\text{odd}}$$

can be factorized as

$$\Pi(s) = As^r \bar{\Pi}_1(s)\bar{\Pi}_2(-s)$$

with $\bar{\Pi}_1(s)$ and $\bar{\Pi}_2(s)$ monic and Hurwitz.

The l_2 case

- Define

$$\Phi_e^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)}$$

$$\Phi_o^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)} \; s^{\mathrm{sgn} A} \; (-1)^{(r-1)/2}$$

- $\Phi^*(s)$ is given by

$$\Phi^*(s) = \begin{cases} \Phi_e^*(s) & r \text{ even} \\ \Phi_o^*(s) & r \text{ odd} \end{cases}$$

The l_2 case

Theorem [Bianchini, Tesi, Vicino 2001]. Assume $\rho < \rho^*$ and $I(\omega) \neq 0 \;\; \forall \omega > 0$.
Then, for sufficiently small $\varepsilon, \delta > 0$, the rational function

$$\Phi(s) = \begin{cases} \Phi_e^*(s)(1+\delta s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1} & \text{for even } r \\ \Phi_o^*(s)\left(\dfrac{s+\varepsilon}{s}\right)^{\mathrm{sgn}A\ (-1)^{(r-1)/2}} \cdot \\ \cdot(1+\delta s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 - \mathrm{sgn}A\ (-1)^{(r-1)/2}} & \text{for odd } r \end{cases}$$

is such that the filter $F(s) = P_0(s)/\Phi(s)$ solves the l_2 RSPR problem.

- The degree of $F(s)$ is bounded by the degree m of $P(s)$

The polyhedral (l_∞, l_1) case

- Non-smooth problem and no closed form solution
- We propose a numerical procedure for the computation of a filter $F(s)$ of given form solving the RSPR problem with guaranteed (possibly maximum) robustness margin $\rho_F \leq \rho^*$.
  - Polynomial filter class of the same degree as $\mathcal{P}_\rho^{(\infty,1)}$

$$F(s;\theta) = P_0(s) + \Delta F(s;\theta)$$

Note. $F(s;0) = P_0(s)$ is a solution for small ρ

- Whether a polynomial $F(s;\theta)$ solving the RSPR problem exists under the only condition $\rho < \rho^*$ is still an open question.

The l_∞ case

- Uncertain polynomial family of degree n (with independent perturbations)

$$\mathcal{P}_\rho^\infty := \left\{ P(s) = P_0(s) + \sum_{i=1}^{n} q_i \hat{a}_{i-1} s^{i-1} : \|q\|_\infty \leq \rho \right\}$$

- Polynomial filter class of degree n

$$F(s;\theta) = P_0(s) + \sum_{i=0}^{n-1} \theta_i s^i$$

$\theta = (\theta_0, \ldots, \theta_{n-1})' \in \mathbb{R}^n$

The l_∞ case

- Define

$$\Phi(s;\theta) = \frac{P_0(s)}{F(s;\theta)}$$

$$\Downarrow$$

$$\gamma_\Phi(\omega;\theta) = \frac{\mathrm{Im}[P_0(j\omega)]X_e(\omega;\theta) - \mathrm{Re}[P_0(j\omega)]X_o(\omega;\theta)}{1 + \mathrm{Re}[P_0(j\omega)]X_e(\omega;\theta) + \mathrm{Im}[P_0(j\omega)]X_o(\omega;\theta)}$$

with

$$X_e(\omega;\theta) = \frac{1}{|P_0(j\omega)|^2} \sum_{i=0,\ i \text{ even}}^{n-1} \theta_i (j\omega)^i$$

$$X_o(\omega;\theta) = \frac{1}{j|P_0(j\omega)|^2} \sum_{i=0,\ i \text{ odd}}^{n-1} \theta_i (j\omega)^i$$

The l_∞ case

- Let

$$\rho_F(\theta)^{-1} = \sup_{\omega\geq 0} \Big(K_e(\omega)\,|\mathrm{Re}[P_0(j\omega)] + \gamma_\Phi(\omega;\theta)\mathrm{Im}[P_0(j\omega)]| + K_o(\omega)\,|\mathrm{Im}[P_0(j\omega)] - \gamma_\Phi(\omega;\theta)\mathrm{Re}[P_0(j\omega)]|\Big)$$

with

$$K_e(\omega) = \frac{1}{|P_0(j\omega)|^2} \sum_{i=0,\ i \text{ even}}^{n-1} \hat{a}_i\omega^i, \qquad K_o(\omega) = \frac{1}{|P_0(j\omega)|^2} \sum_{i=0,\ i \text{ odd}}^{n-1} \hat{a}_i\omega^i$$

Proposition. The filter $F(s;\theta)$ solves the RSPR problem for all $\rho < \rho_F(\theta)$ provided that θ belongs to the set

$$\Theta_{\Phi+} = \left\{\theta \in \mathbb{R}^n \ : \ \inf_{\omega\geq 0} [1 + \mathrm{Re}[P_0(j\omega)]X_e(\omega;\theta) + \mathrm{Im}[P_0(j\omega)]X_o(\omega;\theta)] > 0\right\}$$

The l_∞ case

- The maximum perturbation norm $\rho_F(\theta)$ for which the filter $F(s;\theta)$ solves the RSPR problem can be maximized through the solution of the non-convex optimization problem

$$\theta^* = \arg \sup_{\theta \in \Theta_{\Phi+}} \rho_F(\theta)$$

- $\rho_F(\theta^*)$ is a lower bound to the maximum $\rho \leq \rho^*$ for which a polynomial solution exists
- A performance measure of the filter $F(s;\theta^*)$ is given by the comparison of $\rho_F(\theta^*)$ and ρ^*

The l_1 case

- Uncertain polynomial family

$$\mathcal{P}_\rho^1 := \left\{ P(s) = P_0(s) + \sum_{i=1}^{n} q_i \hat{a}_{i-1} s^{i-1} : \|q\|_1 \leq \rho \right\}$$

- Polynomial filter class

$$F(s;\theta) = P_0(s) + \sum_{i=0}^{n-1} \theta_i s^i$$

$\theta = (\theta_0, \ldots, \theta_{n-1})' \in \mathbb{R}^n$

The l_1 case

- Robustness margin of $F(s;\theta)$

$$\rho_F(\theta)^{-1} = \sup_{\omega \geq 0} \max_{i=0,\ldots,n-1} J_i(\gamma_\Phi(\omega;\theta),\omega)$$

where

$$J_i(\gamma,\omega) = \begin{cases} \dfrac{|\hat{a}_i|\omega^i|\text{Re}[P_0(j\omega)] + \gamma\text{Im}[P_0(j\omega)]|}{|P_0(j\omega)|^2} & i \quad \text{even} \\ \dfrac{|\hat{a}_i|\omega^i|\text{Im}[P_0(j\omega)] - \gamma\text{Re}[P_0(j\omega)]|}{|P_0(j\omega)|^2} & i \quad \text{odd.} \end{cases}$$

- A filter $F(s;\theta^*)$ with guaranteed robustness margin $\rho_F(\theta^*)$ is obtained through the solution of the optimization problem

$$\theta^* = \arg \sup_{\theta \in \Theta_{\Phi+}} \rho_F(\theta)$$

Polyhedral uncertainty: odd (even) perturbations

- Assume only the odd (even) coefficients of the polynomial $P(s)$ are perturbed, i.e., consider the uncertain set

$$\mathcal{P}_\rho^{(\infty,1),e} = \left\{ P(s) \in \mathcal{P}_\rho^{(\infty,1)} \quad : \quad \hat{a}_i = 0, \quad i \text{ even} \right\}$$

or

$$\mathcal{P}_\rho^{(\infty,1),o} = \left\{ P(s) \in \mathcal{P}_\rho^{(\infty,1)} \quad : \quad \hat{a}_i = 0, \quad i \text{ odd} \right\}$$

- The proposed characterization yields a polynomial filter of degree n which is a solution of the RSPR problem for all $\rho < \rho^*$

Polyhedral uncertainty: odd (even) perturbations

Theorem. Let $\rho < \rho^*$. Then, the RSPR problem for the uncertain set $\mathcal{P}_\rho^{(\infty,1),e}$ (resp. $\mathcal{P}_\rho^{(\infty,1),o}$) is solved by the polynomial filter $F_e(s)$ (resp. $F_o(s)$) of degree n given by

$$F_e(s) = \prod_{i=1}^{e_n} (s^2 + 2\zeta\omega_{e,i}s + \omega_{e,i}^2)(1+\delta s)^{n \bmod 2}$$

$$F_o(s) = (s+\varepsilon) \prod_{i=1}^{e_{n-1}} (s^2 + 2\zeta\omega_{o,i}s + \omega_{o,i}^2)(1+\delta s)^{(n-1) \bmod 2}$$

where ζ, δ are sufficiently small scalars, and

$$\Omega_e^0 = \{\omega_{e,1}, \ldots, \omega_{e,e_n}\}, \qquad \Omega_o^0 = \{0, \omega_{o,1}, \ldots, \omega_{o,e_{n-1}}\}$$

are the sets of frequencies at which $\mathrm{Re}[P_0(j\omega)] = 0$ (resp. $\mathrm{Im}[P_0(j\omega)] = 0$).

Conclusion

- Complete frequency domain characterization of the solutions of the robust SPR problem for l_p uncertain polynomial sets
- The proposed characterization yields synthesis methods for the ellipsoidal (l_2) and the polyhedral (l_∞, l_1) case
- The filter output by the above methods has degree bounded by the degree of the uncertain polynomial
- Closed-form rational solution in the ellipsoidal case
- Numerical procedure for the synthesis of polynomial filters with guaranteed robustness margin in the polyhedral case.