A characterization of robust SPR synthesis for systems with parametric uncertainty

Gianni Bianchini
Alberto Tesi
Antonio Vicino

Dipartimento di Sistemi e Informatica, Università di Firenze
Dipartimento di Ingegneria dell’Informazione, Università di Siena

Universita di Siena
A characterization of robust SPR synthesis for systems with $\ell_p$ parametric uncertainty

Outline

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Problem solution for the ellipsoidal ($\ell_2$) case

Frequency domain characterization

Robust SPR vs. $\ell_p$ parametric stability margin

Robust SPR with $\ell_p$ uncertainty: Problem statement

Problem solution for the polyhedral ($\ell_\infty$ and $\ell_1$) case
Motivation and problem statement

Robust absolute stability of nonlinear feedback Lur'ë systems

Siljak 1969

Convergence of recursive identification algorithms [Ljune 1977]

Convergence of recursive identification algorithms [Ljune 1977]
A characterization of robust SPR synthesis for systems with $d \ell$ parametric uncertainty

...contains only Hurwitz polynomials, i.e., the maximum norm of the uncertain parameters such that

$$\mathcal{H} \subseteq \sup_{d \ell} \mathcal{D}$$

Parametric stability margin of the set $\mathcal{D} \ell$

$$0 < d \ell \quad \forall u \in \mathcal{U} \ni (u \ell \cdots u \ell) = b$$

$$\begin{align*}
&u \ell \cdots u \ell = \exists \theta > 0 \quad \forall \theta = \theta \ell \theta \ell \quad \forall \mathcal{H} \ni (s) \theta \ell \\
&\left\{ d \ell \geq \sup_{u \ell} \left\| u \ell \right\| \right\} =: \mathcal{D} \ell
\end{align*}$$

robust SPR problem $d \ell$
Theorem. \[ \text{Anderson et al., 1990} \]

Given the set \( P \), suppose \( \omega < \omega^* \). Then, there exist an integer \( M \) and a Hurwitz polynomial \( R(s) \) of degree \( m + M \) such that the filter \( F(s) = R(s)(s^2 + 1)^M \) solves the \( \mathcal{RSPR} \) problem.

A characterization of robust \( \mathcal{SPR} \) synthesis for systems with \( \mathcal{L}_p \) parametric uncertainty.
Theorem. All rational filters solving the $\mathcal{H}_\infty$ RSPR are given by

\[
\left(\frac{(s)^0 D}{(s)^1 D} - \cdots - \frac{(s)^u D}{(s)^1 D}\right) = (s) \mathcal{G}
\]

Characterization of robust SPR solutions

\[
\begin{align*}
\frac{d}{p} \parallel \cdot \parallel \left< (m) F \right> &\parallel \left< (m) F \right> =: (m) \mathcal{F}/(m) \mathcal{F} \parallel \left< (m) F \right> =: (m) \mathcal{F}/(m) \mathcal{F} \\
\end{align*}
\]

being

\[
0 \leq m \Lambda \frac{d}{I} > \frac{d}{p} \parallel (m) I (m) \mathcal{F}/(m) \mathcal{F} - (m) \mathcal{H} \parallel \ .
\]

I. $(s) \Phi$ is SPR where $(s) \Phi$ is a rational function such that

\[
\frac{(s) \Phi}{(s)^0 D} = (s) \mathcal{D}
\]

Theorem. All rational filters solving the $\mathcal{H}_\infty$ RSPR are given by

\[
\left[\left< (m) \mathcal{F}\right> \parallel \left< (m) \mathcal{F}\right> =: (m) I \right. \left\{ \left< (m) \mathcal{F}\right> \parallel \left< (m) \mathcal{F}\right> =: (m) \mathcal{H} \right. \]

\[
\left(\frac{(s)^0 D}{(s)^1 D} - \cdots - \frac{(s)^u D}{(s)^1 D}\right) = (s) \mathcal{G}
\]

Characterization of robust SPR solutions $\mathcal{H}_\infty$ RSPR
A characterization of robust SPR synthesis for systems with $\ell_p$ parametric uncertainty

\[
\begin{align*}
\mathbb{1}d > \mathbb{2}d = d \\
\mathbb{1}d = d
\end{align*}
\]

\[
\begin{align*}
(m) \gamma > (m) \Phi \gamma > (m) \overline{\gamma} \quad \Leftrightarrow \quad \frac{d}{1} > \frac{d}{p} \| (m) I (m) \Phi \gamma - (m) \mathcal{H} \|
\end{align*}
\]

Phase condition

Characterization of RSPR problem solutions
A characterization of robust SPR synthesis for systems with \( l^p \) parametric uncertainty

\[
\begin{align*}
\frac{d}{p} \| (\omega) I \dot{\omega} - (\omega) H \| & \leq \arg \min (\omega)_* \approx (\omega) \Phi \omega \\
\text{Idea. Find a SPR function such that} & \quad (\omega) \Phi \omega \\
\text{synthesized such that} & \quad \text{solution exists if and only if a SPR function} \\
\text{can be} & \quad (\omega) \Phi \omega \\
\begin{cases}
0 \leq \omega & \quad \frac{d}{p} \| (\omega) I (\omega) \dot{\omega} - (\omega) H \| : (\omega) \dot{\omega} \\
\end{cases} = \frac{d}{p} \omega
\end{align*}
\]

Key. Characterize the set

**RSPR Problem**

\[ d \]
The ellipsoidal ($l_2$) case

- If $\rho < \rho^*$, then a positive real (PR) function $\Phi^*(s)$ exists such that $\gamma_{\Phi^*}(\omega) = \gamma^*(\omega)$.

- $\Phi^*(s)$ can be explicitly computed in closed form via the solution of a polynomial factorization problem.

- The sought function $\Phi(s)$ can be computed by performing a suitable perturbation on $\Phi^*(s)$.

A characterization of robust SPR synthesis for systems with $l_p$ parametric uncertainty.
The case $\frac{1}{2}$ case

Simplifying assumption (can be relaxed). The set $\mathcal{D}$ is such that

\[ 0 < \alpha \text{ and } 0 \neq (\alpha) I \]

The polynomial

\[ \sum_{u=1}^{I} p^\sigma[(s)^2d(s-)^0d] (s-)^2d(s)^0d \]

with \( \Pi \) monic and Hurwitz.

\( (s-)^2\Pi(s)^1 \Pi \forall s = (s) \Pi \)

can be factored as

\[ (s)^2 \Pi(s)^1 \Pi \forall s = (s) \Pi \]
A characterization of robust SPR synthesis for systems with $\ell^p$ parametric uncertainty

\[
\begin{align*}
\text{odd } & \quad (s)^o \Phi \\
\text{even } & \quad (s)^e \Phi
\end{align*}
\]

Define

\[
\forall \nu \in S \quad \frac{(s)^e}{(s)^o} = (s)^e \Phi \\
\frac{(s)^e}{(s)^o} = (s)^e \Phi
\]

The $l^2$ case
The degree of \((s)\Phi\) is bounded by the degree of \((s)P\) if the filter \(\Phi\) solves the \(l2\) RSPR problem.

\[
\begin{align*}
\forall \varepsilon > 0, \exists \mu > 0 : \forall s \in \mathbb{C}, \quad & (s)\Phi(s) > 0, \quad \text{for odd } l, \\
& (s)\Phi(s) < 0, \quad \text{for even } l,
\end{align*}
\]

Then, for sufficiently small \(\varepsilon > 0\), the rational function

\[

\text{Theorem [Bianchi, Tesi, Vicino 2001]}.
\]

Assume and

\[
* d > d^* \quad \text{and} \quad \forall m \neq (m)I.\]

\[d \neq d^* \implies (s)\Phi(s) > 0.\]
The polyhedral \((l_1, l_1)\) case

Whether a polynomial filter class of the same degree as 

\[ d \] is still an open question.

\[ d > d \] is a solution for small 

Note. 

\[ (\theta; s) \mathcal{H} \nabla + (s)^0 \mathcal{H} = (\theta; s) \mathcal{H} \]

- Polynomial filter of given form solving the \(\text{RSPR}\) problem with guaranteed

\[ \mathcal{H} \]

- Possibly maximum robustness margin

We propose a numerical procedure for the computation of a filter

Non-smooth problem and no closed form solution

The polyhedral \((l_1, l_1)\) case
A characterization of robust SPR synthesis for systems with $l^p$ parametric uncertainty

$$u \in \mathbb{R} \ni (1-u \theta, \ldots, 0 \theta) = \theta$$

$$(s \theta \sum_{i=0}^{n} (s)^0 d = (\theta, s)^d)$$

$u$ Polynomial filter class of degree

$$\left\{ d \geq \infty \|b\| : \sum_{i=0}^{n} \left( s \psi \begin{array}{l} \vdots \sum_{i=0}^{n} \psi \end{array} \right) + (s)^0 d = (s)^d \right\} =: \infty \mathcal{d}$$

Perturbations

Uncertain polynomial family of degree $u$ with independent

The case $\infty$
A characterization of robust SPR synthesis for systems with parametric uncertainty

\[ \sum_{i=0}^{\infty} \frac{e^{i \theta \varphi}}{i!} = (\theta \varphi)^{\infty} \]

with

\[ \frac{(\theta \varphi)^{\infty} X[(\varphi)^{\infty}] \Re + (\theta \varphi)^{\infty} X[(\varphi)^{\infty}] \Im + 1}{(\theta \varphi)^{\infty} X[(\varphi)^{\infty}] \Re - (\theta \varphi)^{\infty} X[(\varphi)^{\infty}] \Im} = (\theta \varphi)^{\Phi \infty} \]

Define

The case
\[
\left\{ 0 < \left[ (\theta : \mathfrak{m})^0 X [(\mathfrak{m}^t)^0 P] \right]_{\text{Im}} + (\theta : \mathfrak{m})^2 X [(\mathfrak{m}^t)^0 P] \right\} _{\text{Re}} + 1 \right\} _{\text{Im}} : \mu \notin \Theta \right\} = +\Phi \Theta
\]

Proposition. The filter \( F(\mu) \) solves the RSPR problem for all \( \mu \) belonging to the set \( \mu \in \mathbb{R}^n : \inf_\theta [1 + \text{Re}[P_0(j\theta)] \right\} _{\text{Im}} + \text{Im}[P_0(j\theta)] \right\} _{\text{Re}} + 1 \right\} _{\text{Im}} > 0 \)

Let \( \mu \notin \Theta \)

\[
\sum_{i=u}^{\mu} \frac{|(\mathfrak{m}^t)^0 P|}{1} = (\mathfrak{m})^0 Y
\]

\[
\sum_{i=u}^{\mu} \frac{|(\mathfrak{m}^t)^0 P|}{1} = (\mathfrak{m})^2 Y
\]

with

\[
\left[ [(\mathfrak{m}^t)^0 P] \right] _{\text{Im}} (\theta : \mathfrak{m}) \Phi \nu - [(\mathfrak{m}^t)^0 P] _{\text{Re}} (\mathfrak{m})^0 Y + \\
\left[ [(\mathfrak{m}^t)^0 P] _{\text{Im}} (\theta : \mathfrak{m}) \Phi \nu + [(\mathfrak{m}^t)^0 P] _{\text{Re}} (\mathfrak{m})^2 Y \right] _{\text{Im}} \right\} _{\text{Re}} = \left[ 1 - (\theta)^d d \right]
\]

The case \( \infty \)
The $l_\infty$ case

- The maximum perturbation norm $\rho_F(\theta)$ for which the filter $F(s; \theta)$ solves the RSPR problem can be maximized through the solution of the non-convex optimization problem

$$\theta^* = \arg \sup_{\theta \in \Theta_{\Gamma^+}} \rho_F(\theta)$$

- $\rho_F(\theta^*)$ is a lower bound to the maximum $\rho \leq \rho^*$ for which a polynomial solution exists

- A performance measure of the filter $F(s; \theta^*)$ is given by the comparison of $\rho_F(\theta^*)$ and $\rho^*$.
A characterization of robust SPR synthesis for systems with $l_p$ parametric uncertainty

\[ u^0 \in \bigoplus_{i=1}^{n} (1 - u \theta_1, \ldots, 0 \theta_p) = \theta \]

\[ \sum_{i=1}^{\infty} \theta_i \left( \sum_{i=1}^{\infty} \mu_i s^i \right) + (s)^0 \mathcal{L} = (\theta, s) \mathcal{L} \]

Polynomial filter class \( F(s; \mu) = \mathcal{P}_0(s) + \sum_{i=0}^{\infty} \mu_i s^i \mu_i \)  

Uncertain polynomial family

\[ \left\{ d \geq 1 \| b \| : \sum_{i=-\infty}^{\infty} b_i (s)^i d = (s)^0 d \right\} =: \mathcal{P} \]

Uncertain polynomial family

The \( l_1 \) case
A characterization of robust \( \text{SPR synthesis for systems with } \ell^p \text{ parametric uncertainty} \)

\[
(\theta) \text{d} \max_{\Phi \in \theta} \sup s = * \theta
\]

obtained through the solution of the optimization problem

\[
\begin{cases}
\sup_{\theta \in \Theta} \left| (m')^0 \mathcal{P} \left( \theta \cdot m \right) \right| & \text{odd}, \\
\sup_{\theta \in \Theta} \left| (m')^0 \mathcal{P} \left( \theta \cdot m \right) \right| & \text{even},
\end{cases}
\]

where

\[
(\mu', \mu) \text{ max } \max_{\Phi \in \theta} \sup s = (\theta) \text{d} \sup
\]

Robustness margin of \((\theta \cdot s)' \text{d} \sup \) is

The \( \ell^p \) case

\[
A^* \text{ filter } \text{ with guaranteed robustness margin is } (\theta) \text{d} \sup \]
The proposed characterization yields a polynomial filter of degree $n$ which is a solution of the RSPR problem for all $d > \tilde{d}$.

\[
\left\{ \begin{array}{l}
\text{odd } \tilde{d}, \quad 0 = \tilde{\nu} : (1, \infty)^d \ni (s) d \\
\text{even } \tilde{d}, \quad 0 = \tilde{\nu} : (1, \infty)^d \ni (s) d
\end{array} \right. = (1, \infty)^d d
\]

Assume only the odd (even) coefficients of the polynomial $P(s)$ are perturbed, i.e., consider the uncertain set $P(1, 1) = (s) d$.

Polyhedral uncertainty: odd (even) perturbations
A characterization of robust SPR synthesis for systems with \( p \)-parametric uncertainty.

**Theorem.** Let \( 0 = [(m')^0] \). Then, the RSPR problem for the uncertain set

\[ P(1; 1); \mu \] (resp. \( P(1; 1); \nu \)) is solved by the polynomial filter

\[ F_e(s) = \prod_{i=1}^{n} (s^2 + 2 \nu s + \mu) \mod 2 \]

and

\[ F_o(s) = \prod_{i=1}^{n} (s^2 + 2 \nu s + \mu) \mod 2 \]

where \( \nu, \mu \) are sufficiently small scalars, and

\[ \nu = \frac{1}{2} \sum_{i=1}^{n} (s^2 + 2 \nu s + \mu) \mod 2 \]

A characterization of robust SPR synthesis for systems with \( p \)-parametric uncertainty.

**Theorem.** Let \( d > d \) Then, the RSPR problem for the uncertain set

Polyhedral uncertainty: odd (even) perturbations
A characterization of robust SPR synthesis for systems with \( l^p \) parametric uncertainty

Conclusion

- Guaranteed robustness margin in the polyhedral case.
- Numerical procedure for the synthesis of polynomial filters with
  - Closed-form rational solution in the ellipsoidal case
  - The filter output by the above methods has degree bounded by
    - The degree of the uncertain polynomial of the ellipsoidal \( l^2 \) and the polyhedral \( l^1, l^\infty \) case
- The proposed characterization yields synthesis methods for the
  - Robust SPR problem for \( l^p \) uncertain polynomial sets
- Complete frequency domain characterization of the solutions of

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Conclusion