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Synthesis of restricted complexity controllers for l_2 parametric stability margin maximization

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Outline

- Motivation and problem statement
- l_2 stability margin vs. robust SPR
- A new lower bound
- LMI-based optimization procedure
- Examples
- Conclusion

Motivation and problem statement

- Uncertain SISO plant family

$$\mathcal{P} = \left\{ \begin{array}{c} P(s;\delta) = \dfrac{B_0(s) + \delta' \bar{B}(s)}{A_0(s) + \delta' \bar{A}(s)} \quad ; \quad \delta \in \mathbb{R}^n \\ \\ \bar{B}(s) = [B_1(s) \ldots B_n(s)]' \quad ; \quad \bar{A}(s) = [A_1(s) \ldots A_n(s)]' \end{array} \right\}$$

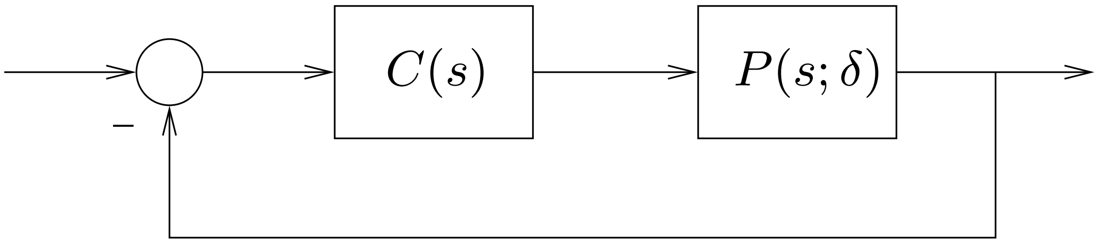
$$\partial B_0 < \partial A_0 \quad ; \quad \partial A_i < \partial A_0 \quad ; \quad \partial B_i < \partial B_0 \quad i = 1, \ldots, n.$$

- Feedback controller

$$C(s) = \frac{N(s)}{D(s)}$$

Motivation and problem statement

- Uncertain control system with ellipsoidal plant uncertainty set



$$\mathcal{B}_\rho = \{\delta \in \mathbb{R}^n \ : \ \|\delta\|_2 \le \rho\}$$

- **Hp**: $C(s)$ stabilizes the nominal plant $P(s;0)$
- l_2 parametric stability margin achieved by $C(s)$

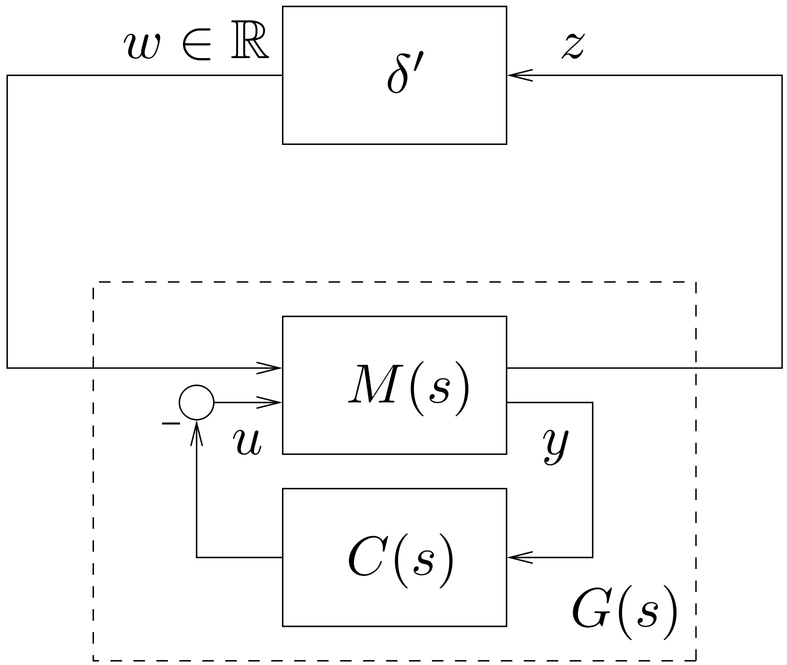
$$\rho_C = \sup \rho$$

$$\text{s.t.}$$

$$\text{the system is stable } \ \forall \delta \in \mathcal{B}_\rho$$

Motivation and problem statement

- Standard rank one control system representation



$$G(s) = -\frac{\bar{A}(s)D(s) + \bar{B}(s)N(s)}{A_0(s)D(s) + B_0(s)N(s)}$$

Motivation and problem statement

- Class of controllers stabilizing the nominal plant

$$\mathcal{C}_0 = \{C(s) \ : \ \text{the interconnection is stable for } \delta = 0\}$$

- l_2 parametric stability margin achieved by $C(s) \in \mathcal{C}_0$

$$\rho_C = \sup \rho$$

$$\text{s.t.}$$

$$1 - \delta' G(j\omega) \neq 0 \ \ \forall \omega \geq 0 \ \ \forall \delta : \|\delta\|_2 < \rho.$$

Motivation and problem statement

SMM problem. Given the uncertain plant family $\mathcal{P}$ and the class of controllers $\mathcal{C}_0$, find $C^*(s) \in \mathcal{C}_0$ such that

$$\rho_{C^*} = \sup_{C(s) \in \mathcal{C}_0} \rho_C$$

- Infinite dimensional convex problem [Rantzer, Megretski, 1994]
  - Approximated solutions (e.g. via Ritz series expansion)
  - High computational complexity and controller order
  - Cannot account for constraints on controller structure

Motivation and problem statement

- Class of restricted complexity controllers

$$\mathcal{C} = \left\{ \begin{array}{c} C_\vartheta(s) = \dfrac{N_\vartheta(s)}{D_\vartheta(s)} = \dfrac{N_0(s) + \vartheta' \bar{N}(s)}{D_0(s) + \vartheta' \bar{D}(s)} \quad : \quad \vartheta \in \Theta \\ \\ \bar{N}(s) = [N_1(s) \dots N_m(s)]' \quad ; \quad \bar{D}(s) = [D_1(s) \dots D_m(s)]' \\ C_0(s) \in \mathcal{C}_0 \end{array} \right\}$$

$$\partial N_0 \leq \partial D_0 \quad ; \quad \partial N_i \leq \partial N_0 \quad ; \quad \partial D_i \leq \partial D_0 \qquad i = 1, \dots, m$$

$$\Theta = \{\vartheta \in \mathbb{R}^m \; : \; Q_j(\vartheta) > 0 \; ; \; j = 1, \dots, k\}$$

Motivation and problem statement

- Example: class of PI compensators

$$\mathcal{C}_{PI} = \left\{ C(s) = K_P + \frac{K_I}{s}, \quad K_P > 0, \quad K_I > 0 \right\}$$

$$C_0(s) = K_P^0 + \frac{K_I^0}{s} \quad \text{(stabilizing)}$$

$$N_0(s) = K_I^0 + K_P^0 s \ ; \ D_0(s) = s \ \ ; \ \ \bar{N}(s) = [1 \ s]' \ ; \ \bar{D}(s) = [0 \ 0]'$$

$$\vartheta = [K_I - K_I^0 \ K_P - K_P^0]'$$

$$\Theta = \left\{ \vartheta \in \mathbb{R}^2 \ : \ Q_1(\vartheta) = \begin{bmatrix} \vartheta_1 + K_I^0 & 0 \\ 0 & \vartheta_2 + K_P^0 \end{bmatrix} > 0 \right\}.$$

Motivation and problem statement

- Possible approach: perform a “projection” of the (infinite order) optimal controller $C^*(s) \in \mathcal{C}_0$ over $\mathcal{C}$

$$C_{\tilde{\vartheta}}(s) = \Pi_{\mathcal{C}}[C^*(s)]$$

  – Must find a notion of Π related to the problem
  – It is difficult to establish how “far” $C_{\tilde{\vartheta}}(s)$ is from the true optimum $C_{\vartheta^*}(s) \in \mathcal{C}$

- Perform a direct optimization over $\mathcal{C}$

Motivation and problem statement

RCSMM problem. Given the uncertain plant family $\mathcal{P}$ and the class of restricted complexity controllers $\mathcal{C}$, find ϑ^* such that $C_{\vartheta^*}(s) \in \mathcal{C}$ achieves the maximum of the closed loop stability margin over the class $\mathcal{C}$, i.e.,

$$\rho_{\vartheta^*} = \sup_{\vartheta \in \Theta} \rho_{\vartheta}.$$

- The RCSMM problem may have local maxima :(

Parametric stability margin vs. robust SPR design

- Characterization of the l_2 parametric stability margin achieved by $C_\vartheta(s) \in \mathcal{C}$.

$$\rho_\vartheta = \sup \rho$$
$$\text{s.t.}$$
$$1 - \delta' G_\vartheta(j\omega) \neq 0 \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 < \rho.$$

where

$$G_\vartheta(s) = -\frac{\bar{A}(s)D_\vartheta(s) + \bar{B}(s)N_\vartheta(s)}{A_0(s)D_\vartheta(s) + B_0(s)N_\vartheta(s)}$$

Parametric stability margin vs. robust SPR design

Theorem.

- The closed loop is stable for all $\|\delta\|_2 < \rho$ if and only if there exists a rational function $\Phi_\vartheta(s)$ such that

$$\Phi_\vartheta(s)[1 - \delta' G_\vartheta(s)] \text{ is } SPR \ \ \forall \delta : \|\delta\|_2 < \rho$$

  [Anderson et al., 1990]

- $\Phi_\vartheta(s)$ can be computed in closed form via the solution of a polynomial factorization problem, and has degree no larger than $2(\partial A_0 + \partial D_0)$ [Bianchini, Tesi, Vicino, 2001].

Parametric stability margin vs. robust SPR design

- Once $\Phi_\vartheta(s)$ is known, the l_2 stability margin is given by

$$\rho_\vartheta = \inf_{\omega \geq 0} r(\vartheta, \omega)$$

where

$$r(\vartheta, \omega) = \frac{1}{\| R_\vartheta(\omega) - \gamma_{\Phi_\vartheta}(\omega) I_\vartheta(\omega) \|_2}$$

being

$$R_\vartheta(\omega) = \mathrm{Re}[G_\vartheta(j\omega)] \ ; \ I_\vartheta(\omega) = \mathrm{Im}[G_\vartheta(j\omega)] \ ; \ \gamma_{\Phi_\vartheta}(\omega) = \frac{\mathrm{Im}[\Phi_\vartheta(j\omega)]}{\mathrm{Re}[\Phi_\vartheta(j\omega)]}$$

A new lower bound

- Perturbed version of $\Phi_\vartheta(s)$ around $\bar{\vartheta} \in \Theta$ with $C_{\bar{\vartheta}}(s)$ stabilizing

$$\Psi_{\bar{\vartheta},\vartheta}(s) = \Phi_{\bar{\vartheta}}(s) \frac{A_0(s)D_\vartheta(s) + B_0(s)N_\vartheta(s)}{A_0(s)D_{\bar{\vartheta}}(s) + B_0(s)N_{\bar{\vartheta}}(s)}$$

(depending affinely on ϑ)

$$\Downarrow$$

- "Approximate" stability margin around $\bar{\vartheta}$

$$\tilde{\rho}(\bar{\vartheta};\vartheta) = \inf_{\omega \geq 0} \tilde{r}(\bar{\vartheta};\vartheta,\omega)$$

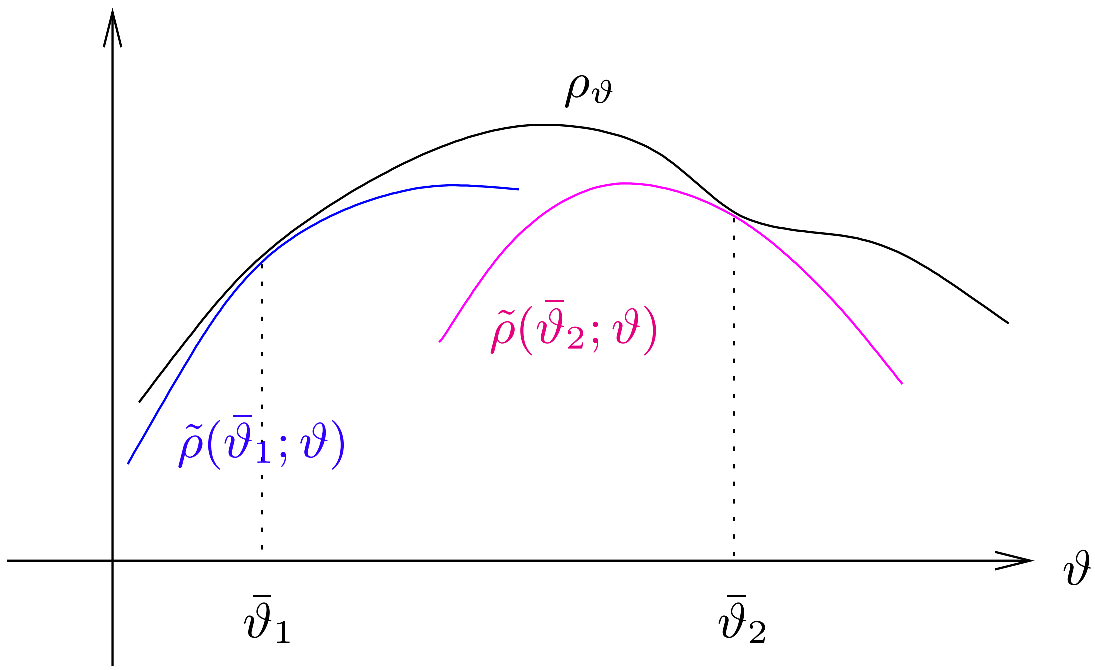
where

$$\tilde{r}(\bar{\vartheta};\vartheta,\omega) = \frac{1}{\|R_\vartheta(\omega) - \gamma_{\Psi_{\bar{\vartheta},\vartheta}}(\omega) I_\vartheta(\omega)\|_2}.$$

A new lower bound

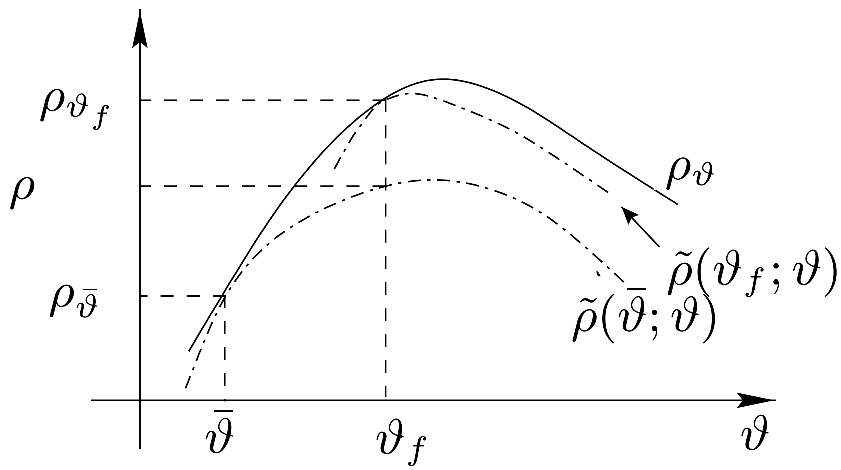
Theorem.

1. $\tilde{\rho}(\bar{\vartheta};\vartheta)$ is a lower bound for ρ_ϑ (tight at $\bar{\vartheta}$)
2. Under generic conditions, $\tilde{\rho}(\bar{\vartheta};\vartheta)$ is a smooth function in a neighbourhood of $\bar{\vartheta}$.



Exploiting the lower bound

- Starting at $\bar{\vartheta}$
  - Find, if possible ϑ_f such that $\tilde{\rho}(\bar{\vartheta};\vartheta_f) \geq \rho$ for some $\rho > \rho_{\vartheta}$ (e.g. use bisection)
  - Set $\bar{\vartheta} = \vartheta_f$ and repeat the process until some optimality criterion is met



- Problem: must find ϑ_f such that $\tilde{\rho}(\bar{\vartheta};\vartheta_f) \geq \rho$ for fixed ρ

LMI-based optimization procedure

Theorem. Given $\bar{\vartheta}$ for which $\rho_{\bar{\vartheta}}$ is well defined (i.e. $C_{\bar{\vartheta}}(s)$ stabilizes the nominal plant), compute $\Phi_{\bar{\vartheta}}(s)$ and consider the corresponding $\Psi_{\bar{\vartheta},\vartheta}(s)$. Given $\rho > 0$, compute a canonical controllable state space realization $[A(\bar{\vartheta}), B, C(\vartheta,\bar{\vartheta},\rho), D(\vartheta,\bar{\vartheta},\rho)]$ of the transfer function

$$T(\Psi_{\bar{\vartheta},\vartheta},\rho,\vartheta;s) = \Psi_{\bar{\vartheta},\vartheta}(s)\begin{bmatrix} I & \rho G_{\vartheta}(s) \\ \rho G'_{\vartheta}(s) & 1 \end{bmatrix}$$

Then, there exists $\vartheta \in \Theta$ such that

$$\tilde{\rho}(\bar{\vartheta};\vartheta) \geq \rho$$

if and only if ϑ is a solution of the following LMI feasibility problem

$$\begin{bmatrix} A'(\bar{\vartheta})X + XA(\bar{\vartheta}) & XB - C'(\vartheta,\bar{\vartheta},\rho) \\ B'X - C(\vartheta,\bar{\vartheta},\rho) & -D(\vartheta,\bar{\vartheta},\rho) - D'(\vartheta,\bar{\vartheta},\rho) \end{bmatrix} < 0$$

$$Q_j(\vartheta) > 0 \quad ; \quad j = 1,\ldots,k$$

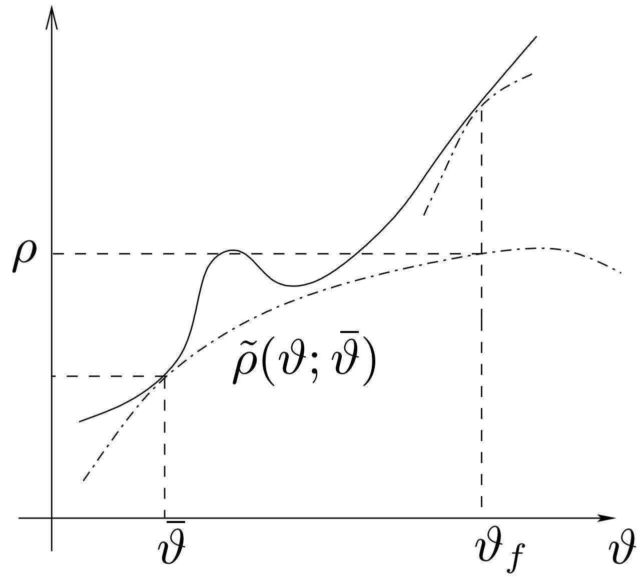
$$X = X' > 0$$

LMI-based optimization procedure

- The RCSMM problem can be solved by means of a procedure involving at each step
  - The computation of a robust SPR filter $\Phi_{\bar{\vartheta}}(s)$ (polynomial factorization problem)
  - The solution of a sequence of LMI feasibility problems (bisection on ρ)
- Properties
  - At each step, $C_{\bar{\vartheta}}(s)$ is a stabilizing controller
  - The sequence $\rho_{\bar{\vartheta}}$ of the stability margins achieved by $C_{\bar{\vartheta}}(s)$ is nondecreasing
- Under generic conditions, the procedure stops at a local maximum of ρ_{ϑ}

LMI-based optimization procedure

- LMI problem dimension
  - At most $(\partial A_0 + \partial D_0)[2(\partial A_0 + \partial D_0) + 1] + m$ parameters
  - Symmetric matrix of dimension $\leq 2(\partial A_0 + \partial D_0) + n + 1$ plus $Q_j(\vartheta), \quad j = 1, \ldots, k$
- Interesting possibile algorithm behaviour



A simple example

- Uncertain plant

$$P(s;\delta) = \frac{1+\delta_1}{s+1+\delta_2}.$$

- The class $\mathcal{P}$ is defined by

$$B_0(s) = 1;\ \ A_0(s) = s+1;\ \ \bar{B}(s) = [1\ 0]';\ \ A_0(s) = [0\ 1]';$$
$$\delta = [\delta_1\ \delta_2]'.$$

- Proportional controller

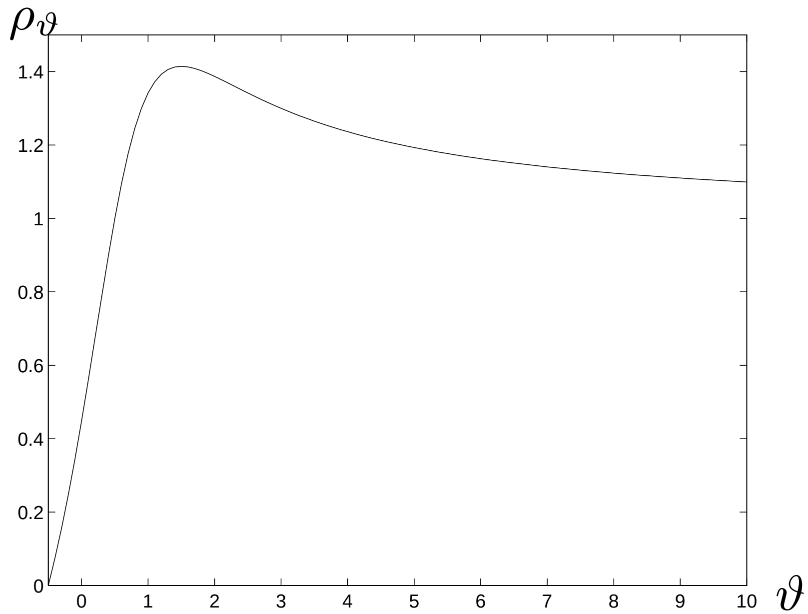
$$C_\vartheta(s) = -0.5 + \vartheta \quad : \quad \vartheta \in \mathbb{R}$$

- The class $\mathcal{C}$ is defined by

$$N_0(s) = -0.5;\ \ D_0(s) = 1;\ \ \bar{N}(s) = 1;\ \ \bar{D}(s) = 0;\ \ \Theta = \mathbb{R}$$

A simple example

- The nominal closed loop is stable for all $\vartheta > -0.5$
- The parametric stability margin is easily computed explicitly



A simple example

- Computation of $\Psi_{\bar{\vartheta},\vartheta}(s)$ for generic $\bar{\vartheta}$

$$\Psi_{\bar{\vartheta},\vartheta}(s) = \frac{s + 0.5 + \vartheta}{s + \varepsilon}.$$

- In this particular case $\Psi_{\bar{\vartheta},\vartheta}(s)$ does not depend on $\bar{\vartheta}$

$$\Downarrow$$

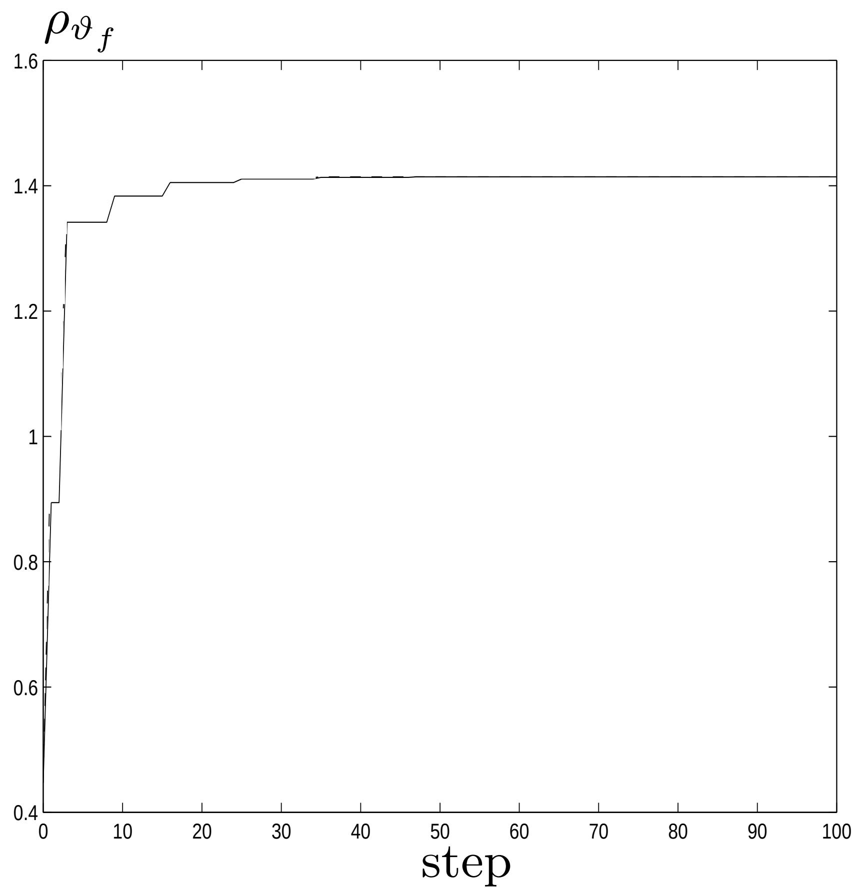
$$\rho_{\vartheta} = \tilde{\rho}(\bar{\vartheta};\vartheta) \quad \forall\bar{\vartheta}$$

$$\Downarrow$$

Convex problem!

A simple example

- Sequence of feasible lower bounds



Another example

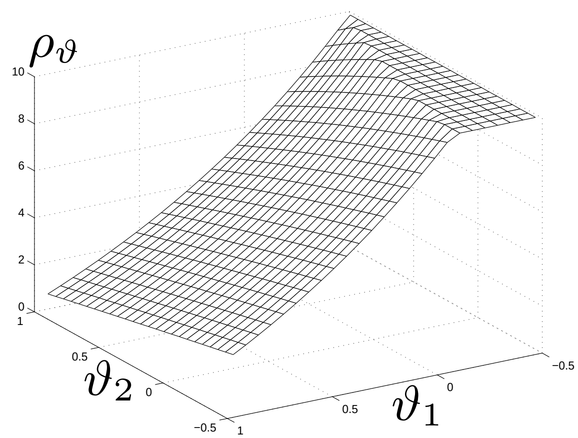
- Uncertain plant

$$P(s;\delta) = \frac{10+\delta_1}{(s+1)(s^2+(2+0.1\delta_2)s+10)}$$

- PI controller

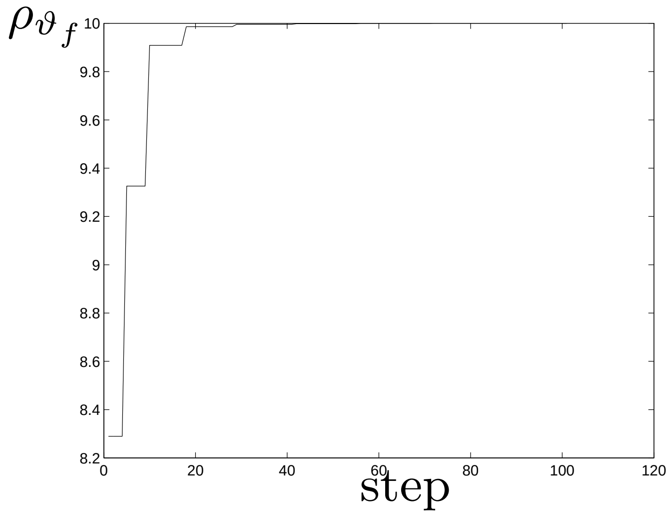
$$C(s;\vartheta) = 1+\vartheta_1+\frac{0.5+\vartheta_2}{s} \quad ; \quad \vartheta\in\mathbb{R}^2$$

- “True” stability margin



Another example

- Sequence of feasible lower bounds



- Optimal PI controller

$$C(s; [-0.54 \quad -0.45]') = 0.46 + \frac{0.05}{s}$$

Conclusion

- l_2 parametric stability margin maximization for uncertain plants with rank one perturbations by means of controllers of fixed structure depending affinely on a tunable parameter vector (RCSMM)
- Optimization based on convex maximization of a suitable lower bound
- Iterative procedure based on robust SPR filter synthesis and LMI feasibility problem solution
- Convergence to local optimum in the numerous examples worked out.

References

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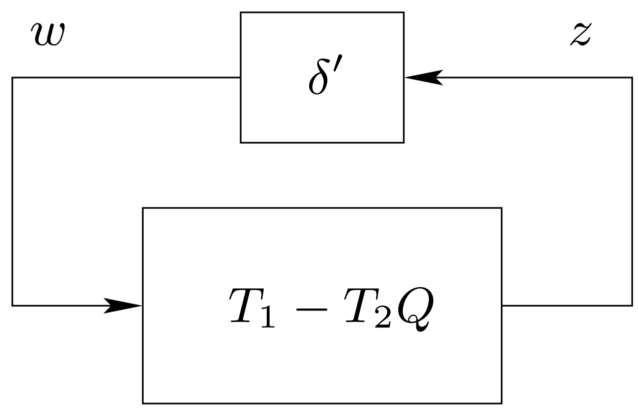
Parametric stability margin vs. robust SPR design



- Robust stability condition

$$\exists Q(s) \in RH_\infty \quad : \quad [1 - \delta'(T_1(s) - T_2(s)Q(s))]^{-1} \in RH_\infty \;\; \forall \delta \in \mathcal{B}_\rho$$

$$\Updownarrow \quad \text{[Anderson et al. 1990]}$$

$$\exists Q(s), \Phi(s) \in RH_\infty \;:\; \Phi(s)[1 - \delta'(T_1(s) - T_2(s)Q(s))] \text{ is SPR } \;\forall \delta \in \mathcal{B}_\rho$$

$$\Updownarrow$$

Parametric stability margin vs. robust SPR design

$$\Updownarrow$$

$$\exists \bar{Q}(s), \Phi(s) \in RH_\infty \ : \ \Phi(s) - \delta'(T_1(s)\Phi(s) - T_2(s)\bar{Q}(s)) \text{ is SPR } \ \forall \delta \in \mathcal{B}_\rho$$

$$\Updownarrow$$

$$\exists \bar{Q}(s), \Phi(s) \in RH_\infty \ :$$

$$\mathrm{Re}[\Phi(j\omega)] - \rho^2 \|\mathrm{Re}[T_1(j\omega)\Phi(j\omega) - T_2(j\omega)\bar{Q}(j\omega)]\|_2^2 \ \ \forall \omega \geq 0$$

$$\Updownarrow$$

$$\exists \bar{Q}(s), \Phi(s) \in RH_\infty \ :$$

$$T(s) = \begin{bmatrix} \Phi(s)I & \rho[T_1(s)\Phi(s) - T_2(s)\bar{Q}(s)] \\ \rho[T_1'(s)\Phi(s) - T_2'(s)\bar{Q}(s)] & \Phi(s) \end{bmatrix} \text{ is SPR}$$

LMI formulation

- Let $\Phi(s)$,$\bar{Q}(s)$ be parametrized affinely, e.g.

$$\Phi(s) = 1 + \sum_{i=1}^{N} \phi_i \left(\frac{s-p}{s+p}\right)^i, \quad \bar{Q}(s) = \sum_{i=0}^{N} q_i \left(\frac{s-p}{s+p}\right)^i$$

- LMI robust stability condition for fixed ρ and N

$$\exists \phi_1, \ldots, \phi_N, q_1 \ldots, q_N, \; X = X' > 0 \quad \text{s.t.}$$

$$\begin{bmatrix} A'X + XA & XB - C'(\phi_i, q_i, \rho) \\ B'X - C(\phi_i, q_i, \rho) & -D(\phi_i, q_i, \rho) - D'(\phi_i, q_i, \rho) \end{bmatrix} > 0$$

where $T(s) \equiv [A, B, C(\phi_i, q_i, \rho), D(\phi_i, q_i, \rho)]$

LMI formulation

- For fixed ρ and N, finding controller parameters such that the stability margin is greater or equal to ρ amounts to the solution of an LMI feasibility problem
- Solving the SMM problem is complicated
  - The dimension of the LMI grows as ρ approaches the optimum
  - The optimal $Q(s)$ and the optimal controller have arbitrarily high degree