A convex lower bound for the real l_2 parametric stability margin of linear control systems with restricted complexity controllers

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***Abstract*— In this paper the problem of restricted complexity stability margin maximization (RCSMM) for single-input single-output (SISO) plants affected by rank one real perturbations is considered. This problem amounts to maximizing the real l_2 parametric stability margin over an assigned class of restricted complexity controllers, which are described by rational transfer functions of fixed order with coefficients depending affinely on some free parameters. It is shown that the RCSMM problem, which is nonconvex in general, can be approached by means of convex optimization methods. Specifically, a lower bound of the stability margin, whose maximization can be accomplished via Linear Matrix Inequality (LMI) techniques, is developed.**

I. Introduction

The parametric stability margin is a typical measure of the robustness of feedback control systems subject to parametric plant uncertainty. In particular, the stability margin maximization (SMM) problem, i.e., the design of controllers maximizing the parametric stability margin, has received considerable attention in the literature (see [1] for a comprehensive reference on this problem).

The most general result is given in [2] and it concerns the class of uncertain systems with rank one real perturbations, which includes single-input single-output (SISO) systems with transfer function coefficients depending affinely on the uncertain parameters. In that paper, Rantzer and Megretski, exploiting the link between robust stability and robust strict positive realness, showed that the optimization problem can be made convex provided that a suitable controller parameterization is employed.

Despite this remarkable result, some issues still deserve to be investigated in order to devise satisfactory design techniques. In particular, since the optimization problem in [2] is infinite dimensional, suboptimal solutions must be looked for through finite dimensional convex programming [3],[4]. Hence, a suitable structure for the approximating solution must be found in order to tackle the computational burden and the complexity of the controller [2],[5].

It is also important to recall that, in the majority of practical applications, it is often mandatory to employ controllers with a prescribed structure, such as PID or lead-lag compensators [6]. In this respect, it is worth to remark that a renewed interest in the synthesis of low-order controllers, mainly based on robust parametric control techniques, has been observed recently [7],[8],[9],[10]. In particular, in [10], an LMI-based method for the design of fixed order controllers based on an inner approximation of the stability domain of polynomials via strictly positive real (SPR) conditions is derived.

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The above considerations naturally lead to the introduction of the restricted complexity stability margin maximization (RCSMM) problem, which amounts to maximizing the parametric stability margin of a family of SISO uncertain plants with rank one real perturbations over a given class of controllers. Such a class of restricted complexity controllers is characterized by fixed order rational transfer functions with coefficients depending affinely on some free design parameters.

Despite its simple formulation, the RCSMM problem discloses several difficulties. In particular, the solution of this problem implicitly calls for an efficient characterization of the restricted complexity controllers which stabilize the nominal plant of the family. This is a very difficult task and few results have been obtained so far (see [7],[11] for the PID case). It is also to remark that in general the RCSMM problem cannot be solved directly by means of convex optimization techniques because of the restricted complexity condition imposed on the controller class.

In this paper we extend in several respects the preliminary results reported in [12],[13],[14] in order to relax the above difficulties. A new characterization of the real l_2 parametric stability margin pertaining to a given stabilizing controller of the restricted complexity class is given. More specifically, on the basis of previous results on the design of filters ensuring robust strict positive realness for affine families of polynomials [15], an SPR rational function is associated to each stabilizing controller. Such a characterization makes it possible to compute a suitable lower bound of the stability margin pertaining to each stabilizing controller. In particular, the maximization of this lower bound, which provides a locally optimal approximation of the stability margin in some neighborhood of the given stabilizing controller, amounts to solving a Generalized Eigenvalue Problem (GEVP) [16].

Notation: $\mathbb{R}^n$: real n-space; $v \in \mathbb{R}^n$: vector of $\mathbb{R}^n$; v': transpose of v; $\|v\|_2$: l_2 norm of v; $\mathbf{A} \in \mathbb{R}^{n \times m}$: real $n \times m$ matrix; $\mathbf{A} > 0$: matrix $\mathbf{A}$ is positive definite; $\mathbb{C}$: the complex plane; $s \in \mathbb{C}$: complex number; $\mathrm{Re}[s], \mathrm{Im}[s]$: real and imaginary parts of s; $\pi(s)$: a polynomial in the complex variable s; $\partial\pi$: degree of a polynomial $\pi(s)$; $P(s)$: rational function of s; $\bar{M}(s)$: rational function vector/matrix.

II. Problem formulation and preliminaries

Consider the following class of uncertain SISO plants

$$\mathcal{P} = \left\{ P(s;\delta) = \frac{B_0(s) + \delta' \bar{B}(s)}{A_0(s) + \delta' \bar{A}(s)} \quad : \quad \delta \in \mathbb{R}^n \right\} \qquad (1)$$

where $\bar{B}(s) = [B_1(s) \dots B_n(s)]'$, $\bar{A}(s) = [A_1(s) \dots A_n(s)]'$, $\delta = [\delta_1 \dots \delta_n]'$ is the uncertain parameter vector and $B_i(s), A_i(s)$, $i = 0, \dots, n$ are given polynomials. In the sequel the following simplifying assumption on the degree of the polynomials involved in (1) will be enforced.

Assumption 1: (Plant degree constraint)

$$\partial B_0 < \partial A_0 \quad ; \quad \partial A_i < \partial A_0, \quad \partial B_i < \partial B_0, \quad i = 1, \dots, n.$$

The above assumption ensures that $P(s;\delta)$ is strictly proper and its order is invariant with respect to δ.

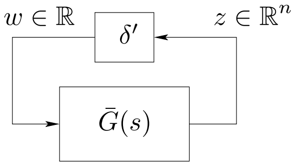

Fig. 1. Closed loop LFT representation.

Hereafter, we denote by $P_\delta \in \mathcal{P}$ the plant with transfer function $P(s;\delta)$. In particular, P_0 will be referred to as the *nominal plant* of the family. We denote by $\mathcal{S}_0$ the set of controllers stabilizing P_0.
Consider the feedback interconnection of the uncertain plant P_δ and a controller $C \in \mathcal{S}_0$ with transfer function

$$C(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are given polynomials. A standard representation of the resulting feedback control loop can be obtained via a linear fractional transformation (LFT) in the form reported in Fig. 1, where

$$\bar{G}(s) = -\frac{D(s)\bar{A}(s) + N(s)\bar{B}(s)}{D(s)A_0(s) + N(s)B_0(s)}.$$

Since the perturbation δ is a vector (i.e., w is a scalar), this representation is usually said to be *rank one*.
We recall the notion of real l_2 parametric stability margin [1].

Definition 1: Given the uncertain plant class $\mathcal{P}$ and a controller $C \in \mathcal{S}_0$, the real l_2 parametric stability margin ρ_C pertaining to such controller is defined as the maximal ρ such that the closed loop is stable for all $\|\delta\|_2 < \rho$.
We have the following well-known expression for ρ_C.

Lemma 1: Let $\mathcal{P}$ and $C \in \mathcal{S}_0$ be given. Then,

$$\rho_C = \sup \rho$$
$$\text{s.t.} \quad 1 - \delta'\bar{G}(j\omega) \neq 0, \quad \forall \omega \geq 0 \quad \forall \delta : \|\delta\|_2 < \rho.$$

One of the most investigated robust control problems is the parametric stability margin maximization (SMM) which amounts to finding the controller which maximizes ρ_C over the class of stabilizing controllers $\mathcal{S}_0$ [1],[2].
Despite the fundamental results given in [2], where the SMM problem is shown to be convex with respect to a suitable parameterization of robustly stabilizing controllers, the actual computation of the optimal controller shows practical difficulties, especially when constraints on the controller structure are imposed, a very common situation in practical applications.
Motivated by the above consideration, we consider the SMM problem when the controller is assumed to have a given structure. In particular, we introduce the following class of *restricted complexity controllers*:

$$\mathcal{C} = \left\{ C(s;\vartheta) = \frac{N(s;\vartheta)}{D(s;\vartheta)} = \frac{N_0(s) + \vartheta'\bar{N}(s)}{D_0(s) + \vartheta'\bar{D}(s)} \ : \ \vartheta \in \Theta \right\} \tag{2}$$

where $\bar{N}(s) = [N_1(s) \dots N_m(s)]'$, $\bar{D}(s) = [D_1(s) \dots D_m(s)]'$, $N_i, D_i(s)$, $i = 0, \dots, m$ are given polynomials, $\vartheta = [\vartheta_1 \dots \vartheta_m]' \in \mathbb{R}^m$ is a free controller parameter vector, $\Theta \subseteq \mathbb{R}^m$ is of the form

$$\Theta = \{\vartheta \in \mathbb{R}^m \ : \ \mathbf{Q}_j(\vartheta) > 0 \ ; \ j = 1, \dots, k\}$$

being $\mathbf{Q}_j(\vartheta)$, $j = 1, \dots, k$ symmetric matrices whose entries depend affinely on ϑ. The choice of the controller class $\mathcal{C}$ in (2) is motivated mainly by practical reasons, since many widely used structures such as PID or lag-lead can be seen as members of such a class. The choice of the structure of the admissible parameter set Θ is motivated by the fact that this structure enters naturally into LMI problems and is able to model several useful constraints on the controller, e.g., coefficient positivity.
We refer to C_ϑ as the controller with transfer function $C(s;\vartheta)$. In particular, without loss of generality, C_0 is assumed to be a stabilizing controller as stated next.

Assumption 2: C_0 stabilizes P_0, i.e., $C_0 \in \mathcal{S}_0$.
Also, to avoid excessive technicalities, the following condition on the polynomials involved in (2) is enforced.

Assumption 3: (Controller degree constraint)

$$\partial N_0 \leq \partial D_0, \quad \partial N_i \leq \partial N_0, \quad \partial D_i < \partial D_0, \quad i = 1, \dots, m.$$

The above assumption ensures that $C(s;\vartheta)$ is proper and that its order is invariant with respect to ϑ.
The problem of stability margin maximization over the class of restricted complexity controllers $\mathcal{C}$ can be stated as follows, where $\rho(\vartheta)$ denotes the parametric stability margin achieved by C_ϑ according to Definition 1, i.e., $\rho(\vartheta) = \rho_{C_\vartheta}$.

RCSMM problem. Given the plant family $\mathcal{P}$ and the class of restricted complexity controllers $\mathcal{C}$, find the parameters $\vartheta^* \in \Theta$ such that the controller $C_{\vartheta^*} \in \mathcal{C}$ achieves the maximum of the closed loop stability margin over the class $\mathcal{C}$, i.e.,

$$\rho_{C_{\vartheta^*}} = \sup_{\vartheta \in \Theta} \rho(\vartheta). \tag{3}$$

III. Main results

Several difficulties arise in solving the RCSMM problem. In particular, in view of Definition 1, it is to observe that $\rho(\vartheta)$ is well defined only for ϑ belonging to the set $\Theta_{stab} \cap \Theta$ where

$$\Theta_{stab} = \{\vartheta \in \mathbb{R}^m \ : \ C_\vartheta \in \mathcal{S}_0\}.$$

Therefore, solving (3) implicitly calls for an efficient characterization of Θ_{stab}. Unfortunately, this is a very difficult task and few results have been presented so far in the literature (see, e.g., [7],[11]). It is also to observe that the RCSMM problem cannot be solved in general via convex techniques because of the restricted complexity condition on the controller. Indeed, it is not difficult to find examples which display local maxima.
In this paper, the RCSMM problem is approached in order to relax the above difficulties. First, in Subsection III-A a new characterization of the parametric stability margin $\rho(\vartheta)$ pertaining to C_ϑ, $\vartheta \in \Theta_{stab}$ is developed. More specifically, exploiting earlier results on Strict Positive Realness (SPR) [15], a suitable SPR rational function is associated to each C_ϑ, $\vartheta \in \Theta_{stab}$, thus providing an implicit parameterization of the set Θ_{stab}. Subsequently, on the basis of such characterization, a lower bound of $\rho(\vartheta)$ is developed in Subection III-B. In particular, this bound provides a locally optimal approximation of $\rho(\vartheta)$ in some neighborhood of a given controller parameter vector, and its maximization can be accomplished via LMI techniques.

A. Characterization of the real l_2 parametric stability margin

The stability margin $\rho(\vartheta)$ pertaining to C_ϑ, $\vartheta \in \Theta_{stab}$, can be computed in accordance with Lemma 1 once $\bar{G}(s)$ is replaced with

$$\bar{G}(s;\vartheta) = -\frac{D(s;\vartheta)\bar{A}(s) + N(s;\vartheta)\bar{B}(s)}{D(s;\vartheta)A_0(s) + N(s;\vartheta)B_0(s)}.$$

However, we look for a special characterization of $\rho(\vartheta)$ which is obtained by associating to C_ϑ a suitable SPR rational function. To proceed, let us rewrite $\bar{G}(s;\vartheta)$ as

$$\bar{G}(s;\vartheta) = -\left[\frac{\pi_1(s;\vartheta)}{\pi_0(s;\vartheta)}, \ldots, \frac{\pi_n(s;\vartheta)}{\pi_0(s;\vartheta)}\right]' \quad (4)$$

where, according to the expressions in (2), the polynomials $\pi_i(s;\vartheta)$, $i = 0, \ldots, n$ are given by

$$\pi_i(s;\vartheta) = D_0(s)A_i(s) + N_0(s)B_i(s) + \vartheta'[\bar{D}(s)A_i(s) + \bar{N}(s)B_i(s)]. \quad (5)$$

Note that, since C_ϑ is assumed to stabilize the nominal plant, the polynomial $\pi_0(s;\vartheta)$ is Hurwitz. From (4) it follows that the characteristic polynomial of the closed loop system has the expression

$$\pi(s;\delta;\vartheta) = \pi_0(s;\vartheta) + \sum_{i=1}^{n} \delta_i \pi_i(s;\vartheta). \quad (6)$$

It is not difficult to check that Assumption 1 and Assumption 3 ensure that the degree of the characteristic polynomial is invariant with respect to δ and ϑ, specifically $\partial\pi = \partial D_0 + \partial A_0$.

Let us now introduce the two functions

$$\bar{R}(\omega;\vartheta) = \mathrm{Re}[\bar{G}(j\omega;\vartheta)] \quad ; \quad \bar{I}(\omega;\vartheta) = \mathrm{Im}[\bar{G}(j\omega;\vartheta)],$$

the related set

$$\Omega_\vartheta = \left\{\omega \geq 0 \quad : \quad \bar{I}(\omega;\vartheta) = 0\right\}, \quad (7)$$

and the polynomial

$$\Pi(s;\vartheta) = \textstyle\sum_{i=1}^{n} \pi_0(s;\vartheta)\pi_i(-s;\vartheta) \cdot \cdot \left[\pi_0(-s;\vartheta)\pi_i(s;\vartheta) - \pi_0(s;\vartheta)\pi_i(-s;\vartheta)\right].$$

It can be verified that for each ϑ, any nonzero frequency $\hat{\omega} \in \Omega_\vartheta$ must be a common root of n polynomials in ω, which implies that the existence of such $\hat{\omega}$ is not generic especially for large n. Therefore, the assumption $\Omega_\vartheta = \{0\}$ will be enforced in the sequel, as it holds almost everywhere in the controller parameter space. Under this assumption, it can be shown that $\Pi(s;\vartheta)$ is such that $\Pi(j\omega;\vartheta) \neq 0$ for all $\omega > 0$ and therefore the following factorization holds (see also Lemma 7 in [15]):

$$\Pi(s;\vartheta) = \alpha_\vartheta s^{q_\vartheta} \Pi_1(s;\vartheta)\Pi_2(-s;\vartheta) \quad (8)$$

where α_ϑ is a real constant, $q_\vartheta \geq 1$ is an integer and $\Pi_1(s;\vartheta)$, $\Pi_2(s;\vartheta)$ are uniquely determined monic Hurwitz polynomials. The following result provides the sought characterization of $\rho(\vartheta)$.

Theorem 1: Let $\vartheta \in \Theta_{stab}$ and suppose $\Omega_\vartheta = \{0\}$. Introduce the rational function

$$\Phi(s;\vartheta;\varepsilon,\tau) = \begin{cases} \dfrac{\Pi_1(s;\vartheta)}{\Pi_2(s;\vartheta)}(1+\tau s)^{\partial\Pi_2 - \partial\Pi_1} & \text{for even } q_\vartheta \\[2ex] \dfrac{\Pi_1(s;\vartheta)}{\Pi_2(s;\vartheta)}(s+\varepsilon)^{\xi_1} \cdot \\ \cdot(1+\tau s)^{\xi_2} & \text{for odd } q_\vartheta \end{cases}$$

where ε and τ are positive scalars, and

$$\xi_1 = \mathrm{sgn}[\alpha_\vartheta(-1)^{(q_\vartheta-1)/2}]$$

$$\xi_2 = \partial\Pi_2 - \partial\Pi_1 - \mathrm{sgn}[\alpha_\vartheta(-1)^{(q_\vartheta-1)/2}],$$

being $\Pi_1(s;\vartheta)$, $\Pi_2(s;\vartheta)$, α_ϑ, q_ϑ as in (8). Then, for all ρ such that $0 < \rho < \rho(\vartheta)$, there exist $\varepsilon > 0$ and $\tau > 0$ such that

1) $\Phi(s;\vartheta;\varepsilon,\tau)$ is SPR
2) The following inequality holds

$$\rho < \rho_\Phi(\vartheta;\varepsilon,\tau) \leq \rho(\vartheta) \quad (9)$$

where

$$\rho_\Phi(\vartheta;\varepsilon,\tau) = \inf_{\omega \geq 0} r_\Phi(\omega;\vartheta;\varepsilon,\tau) \quad (10)$$

being

$$r_\Phi(\omega;\vartheta;\varepsilon,\tau) = \|\bar{R}(\omega;\vartheta) - \gamma_\Phi(\omega;\vartheta;\varepsilon,\tau)\bar{I}(\omega;\vartheta)\|_2^{-1} \quad (11)$$

and

$$\gamma_\Phi(\omega;\vartheta;\varepsilon,\tau) = \frac{\mathrm{Im}[\Phi(j\omega;\vartheta;\varepsilon,\tau)]}{\mathrm{Re}[\Phi(j\omega;\vartheta;\varepsilon,\tau)]}.$$

Proof: See Appendix. ■

The above Theorem provides a new characterization of the real l_2 parametric stability margin $\rho(\vartheta)$ pertaining to a controller C_ϑ, $\vartheta \in \Theta_{stab}$. The key element is the SPR rational function $\Phi(s;\vartheta;\varepsilon,\tau)$, which is computed via the polynomial factorization (8) and the appropriate selection of the parameters ε and τ. In this respect, some remarks are in order.

Remark 1: The proof of Theorem 1 makes it clear that, for sufficiently small ε, τ, $\Phi(s;\vartheta;\varepsilon,\tau)$ is SPR and $\rho_\Phi(\vartheta;\varepsilon,\tau)$ is a quite good approximation of $\rho(\vartheta)$. By taking into account the role of ε ($\Pi(s;\vartheta)$ has a singularity at $s = 0$) and τ ($\Phi(s;\vartheta;\varepsilon,0)$ may not be biproper), it turns out that ε (resp. $1/\tau$) should be chosen at least one decade smaller (resp. larger) than all the singularities of $\Pi_1(s;\vartheta)$ and $\Pi_2(s;\vartheta)$.

Remark 2: It is not difficult to show that the order of $\Phi(s;\vartheta;\varepsilon,\tau)$ is related to that of the plant class $\mathcal{P}$ in (1) and the controller class $\mathcal{C}$ in (2). Indeed, exploiting (5), (6), and (8), it turns out that the order of $\Phi(s;\vartheta;\varepsilon,\tau)$ is no larger than $2(\partial A_0 + \partial D_0)$ (see [15]).

Remark 3: The assumption $\Omega_\vartheta = \{0\}$ ensures continuity of the real l_2 parametric stability margin $\rho(\vartheta)$ [1],[17]. The violation of this assumption, which can be readily detected according to the comments related to (7)-(8), makes it possible that the stability margin $\rho(\vartheta)$ is discontinuous at some point $\vartheta = \vartheta_d$. Such a discontinuity issue has been investigated in several papers and regularity results have been derived [18], [19], [20], [1],[17], also exploiting the link with the computation of the structured singular value μ for rank-one uncertainty [18], [19], [20]. Finally, we note that an expression for an SPR $\Phi(s;\vartheta;\varepsilon,\tau)$ can be obtained also in the non-generic case of $\Omega_\vartheta \neq \{0\}$.

B. An LMI lower bound to the real l_2 parametric stability margin

The characterization in Subsection III-A provides an implicit parameterization of Θ_{stab}. However, the maximization of the approximating quantity $\rho_\Phi(\vartheta;\varepsilon,\tau)$ with respect to $\vartheta \in \Theta_{stab} \cap \Theta$ cannot be solved by convex optimization methods directly. In the sequel, a suitable lower bound of $\rho(\vartheta)$, whose maximization can be carried out by means of a standard GEVP, is developed.

To proceed, let $\theta \in \Theta_{stab}$ be given, and consider the corresponding closed loop nominal characteristic polynomial $\pi_0(s;\theta)$ in (6). Let $\Phi(s;\theta;\varepsilon,\tau)$ be computed as in Theorem 1 for some positive ε,τ and consider the rational function

$$\Psi(s;\vartheta;\theta;\varepsilon,\tau) = \Phi(s;\theta;\varepsilon,\tau)\frac{\pi_0(s;\vartheta)}{\pi_0(s;\theta)} \tag{12}$$

which depends affinely on ϑ and obviously satisfies $\Psi(s;\theta;\theta;\varepsilon,\tau) = \Phi(s;\theta;\varepsilon,\tau)$. Moreover, since $\Phi(s;\theta;\varepsilon,\tau)$ is SPR, it follows that also $\Psi(s;\vartheta;\theta;\varepsilon,\tau)$ is SPR for all ϑ in some neighborhood of θ. In this respect, we have the following result.

Lemma 2: Let $\Theta_\theta = \{\vartheta \in \Theta : \ \Psi(s;\vartheta;\theta;\varepsilon,\tau) \text{ is SPR } \}$. Then, $\Theta_\theta \subseteq \Theta_{stab}$.

Proof: We first observe that, since $\Psi(s;\vartheta;\theta;\varepsilon,\tau)$ depends affinely on ϑ, Θ_θ is a convex set. Since $\Phi(s;\theta;\varepsilon,\tau)$ is SPR, from (12) we have that for any $\vartheta \in \Theta_\theta$ the following inequality holds

$$\begin{aligned}&|\arg[\pi_0(j\omega;\vartheta)] - \arg[\pi_0(j\omega;\theta)]| = \\ &|\arg[\Psi(j\omega;\vartheta;\theta;\varepsilon,\tau)] - \arg[\Phi(j\omega;\theta;\varepsilon,\tau)]| < \pi, \quad \forall\ \omega \geq 0.\end{aligned}$$

Recalling that the degree of $\pi_0(s;\theta)$ is invariant with respect to ϑ, from the Bounded Phase Lemma [1] we have that all the polynomials of the following one-parameter family $\{(1-\lambda)\pi_0(s;\theta) + \lambda\pi_0(s;\vartheta),\ \lambda \in [0,1]\}$ have the same stability properties. Hence, the proof follows from Hurwitzness of $\pi_0(s;\theta)$ and convexity of Θ_θ. ∎

The above lemma ensures that, for any $\vartheta \in \Theta_\theta$, the rational function

$$\gamma_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau) = \frac{\text{Im}[\Psi(j\omega;\vartheta;\theta;\varepsilon,\tau)]}{\text{Re}[\Psi(j\omega;\vartheta;\theta;\varepsilon,\tau)]}$$

is bounded for all $\omega \geq 0$. As a consequence, we can define the function of ϑ

$$\rho_\Psi(\vartheta;\theta;\varepsilon,\tau) = \inf_{\omega\geq 0} r_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau) \tag{13}$$

where

$$r_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau) = \|\bar{R}(\omega;\vartheta) - \gamma_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau)\bar{I}(\omega;\vartheta)\|_2^{-1}. \tag{14}$$

Note that, according to (12), the dependence of $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ on ε and τ is inherited by the computation of $\Phi(s;\theta;\varepsilon,\tau)$ as in Theorem 1 (see also Remark 1). The result that follows relates $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ with the stability margin $\rho(\vartheta)$ and its approximating quantity $\rho_\Phi(\vartheta;\varepsilon,\tau)$.

Lemma 3: Let $\theta \in \Theta_{stab}$. Then, the following properties hold for the function $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$.

1) $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ is a lower bound of $\rho(\vartheta)$, i.e.,
$$\rho_\Psi(\vartheta;\theta;\varepsilon,\tau) \leq \rho(\vartheta) \quad \forall\vartheta \in \Theta_\theta. \tag{15}$$
2) $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ is equal to $\rho_\Phi(\vartheta;\varepsilon,\tau)$ for $\vartheta = \theta$, i.e.,
$$\rho_\Psi(\theta;\theta;\varepsilon,\tau) = \rho_\Phi(\theta;\varepsilon,\tau). \tag{16}$$

Proof: Lemma 2 ensures that both $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ and $\rho(\vartheta)$ are well defined for any $\vartheta \in \Theta_\theta$. Moreover, observe that, for fixed θ, ε, τ, both $\gamma_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau)$ and $r_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau)$ are continuous functions of ω for all $\omega \geq 0$, and it holds that

$$r_\Psi(0;\vartheta;\theta;\varepsilon,\tau) = \|\bar{R}(0;\vartheta)\|_2^{-1} \triangleq \rho_0(\vartheta). \tag{17}$$

Since

$$\gamma_\Phi(\omega;\vartheta;0,0) = \arg\min_\gamma \|\bar{R}(\omega;\vartheta) - \gamma\bar{I}(\omega;\vartheta)\|_2 \quad \forall\omega > 0 \tag{18}$$

we get

$$r_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau) \leq r_\Phi(\omega;\vartheta;0,0) \quad \forall\omega > 0. \tag{19}$$

From (17), (19) and continuity of $r_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau)$ with respect to ω for all $\omega \geq 0$, we get that (15) holds.

Finally, condition 2. directly follows from (12). ∎

Condition 2 of Lemma 3 ensures that $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ provides a quite good approximation of $\rho(\vartheta)$ in some neighborhood of θ for sufficiently small values of ε and τ. Moreover, it turns out that the maximization of $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ amounts to solving a GEVP.

Consider the rational matrix

$$\bar{T}(s;\vartheta;\theta;\rho;\varepsilon,\tau) = \Psi(s;\vartheta;\theta;\varepsilon,\tau)\begin{bmatrix} \mathbf{I}_{n\times n} & \rho\bar{G}(s;\vartheta) \\ \rho\bar{G}'(s;\vartheta) & 1 \end{bmatrix}. \tag{20}$$

It turns out that $\bar{T}(s;\vartheta;\theta;\rho;\varepsilon,\tau)$ depends affinely on ϑ for fixed ρ. In particular, the poles of $\bar{T}(s;\vartheta;\theta;\rho;\varepsilon,\tau)$ do not depend on ϑ. This observation suggests that a canonical controllable state space realization of $\bar{T}(s;\vartheta;\theta;\rho;\varepsilon,\tau)$ has the form

$$[\mathbf{A}(\theta;\varepsilon,\tau), \mathbf{B}, \mathbf{C}(\vartheta;\theta;\rho;\varepsilon,\tau), \mathbf{D}(\vartheta;\theta;\rho;\varepsilon,\tau)] \tag{21}$$

where $\mathbf{C}(\vartheta;\theta;\rho;\varepsilon,\tau)$ and $\mathbf{D}(\vartheta;\theta;\rho;\varepsilon,\tau)$ are of the form

$$\begin{aligned}\mathbf{C}(\vartheta;\theta;\rho;\varepsilon,\tau) &= \mathbf{C}_0(\vartheta;\theta;\varepsilon,\tau) + \rho\mathbf{C}_\rho(\vartheta;\theta;\varepsilon,\tau)\\ \mathbf{D}(\vartheta;\theta;\rho;\varepsilon,\tau) &= \mathbf{D}_0(\vartheta;\theta;\varepsilon,\tau) + \rho\mathbf{D}_\rho(\vartheta;\theta;\varepsilon,\tau).\end{aligned} \tag{22}$$

Obviously, once θ is given, the matrices $\mathbf{A}(\theta;\varepsilon,\tau)$ and $\mathbf{B}$ and the affine functions of ϑ $\mathbf{C}_0(\vartheta;\theta;\varepsilon,\tau)$, $\mathbf{C}_\rho(\vartheta;\theta;\varepsilon,\tau)$, $\mathbf{D}_0(\vartheta;\theta;\varepsilon,\tau)$, $\mathbf{D}_\rho(\vartheta;\theta;\varepsilon,\tau)$ can be computed explicitly.

The following result provides the sought maximization of $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ with respect to $\vartheta \in \Theta_\theta \cap \Theta$ by means of the solution of a GEVP.

Theorem 2: Let $\theta \in \Theta_{stab} \cap \Theta$. Then,

$$\max_{\vartheta\in\Theta_\theta\cap\Theta} \rho_\Psi(\vartheta;\theta;\varepsilon,\tau) = \min_{\mu,\mathbf{X},\vartheta} \mu \tag{23}$$

$$\text{s.t.} \quad \begin{aligned}&\mathcal{B}(\mathbf{X},\vartheta;\theta;\varepsilon,\tau) > 0\\ &\mu\mathcal{B}(\mathbf{X},\vartheta;\theta;\varepsilon,\tau) - \mathcal{A}(\mathbf{X},\vartheta;\theta;\varepsilon,\tau) > 0\\ &\mathbf{X} = \mathbf{X}' > 0\end{aligned} \tag{24}$$

$$\text{and} \quad \mathbf{Q}_j(\vartheta) > 0 \quad ; \quad j = 1,\ldots,k \tag{25}$$

where

$$\mathcal{B}(\mathbf{X},\vartheta;\theta;\varepsilon,\tau) = -\begin{bmatrix} \mathbf{A}'(\theta;\varepsilon,\tau)\mathbf{X}+\mathbf{X}\mathbf{A}(\theta;\varepsilon,\tau) & \mathbf{X}\mathbf{B}-\mathbf{C}_0'(\vartheta;\theta;\varepsilon,\tau) \\ \mathbf{B}'\mathbf{X}-\mathbf{C}_0(\vartheta;\theta;\varepsilon,\tau) & -\mathbf{D}_0(\vartheta;\theta;\varepsilon,\tau)-\mathbf{D}_0'(\vartheta;\theta;\varepsilon,\tau) \end{bmatrix} \quad (26)$$

$$\mathcal{A}(\mathbf{X},\vartheta;\theta;\varepsilon,\tau) = \begin{bmatrix} 0 & -\mathbf{C}_\rho'(\vartheta;\theta;\varepsilon,\tau) \\ -\mathbf{C}_\rho(\vartheta;\theta;\varepsilon,\tau) & -\mathbf{D}_\rho(\vartheta;\theta;\varepsilon,\tau)-\mathbf{D}_\rho'(\vartheta;\theta;\varepsilon,\tau) \end{bmatrix} \quad (27)$$

Proof: We first prove that

$$\max_{\vartheta\in\Theta_\theta} \rho_\Psi(\vartheta;\theta;\varepsilon,\tau) = \max_{\rho,\vartheta} \rho \quad (28)$$

$$\text{s.t.} \quad \bar{T}(s;\vartheta;\theta;\rho;\varepsilon,\tau) \text{ in (20) is SPR.}$$

Indeed, for any $\rho > 0$ and any $\vartheta \in \Theta_\theta$, by (13)-(14) it turns out that $\rho < \rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ if and only if

$$\|\bar{R}(\omega;\vartheta) - \gamma_\Psi(\omega;\vartheta;\theta;\varepsilon,\tau)\bar{I}(\omega;\vartheta)\|_2 < \frac{1}{\rho} \quad \forall\omega \geq 0$$

which by strict positive realness of $\Psi(s;\vartheta;\theta;\varepsilon,\tau)$ is equivalent to

$$\mathrm{Re}[\Psi(j\omega;\vartheta;\theta;\varepsilon,\tau)] - \|\mathrm{Re}[\rho\bar{G}(j\omega;\vartheta)\Psi(j\omega;\vartheta;\theta;\varepsilon,\tau)]\|_2 > 0 \ \forall\omega \geq 0. \quad (29)$$

In turn, by a standard Schur complement argument on the matrix function $\mathrm{Re}[\bar{T}(j\omega;\vartheta;\theta;\rho;\varepsilon,\tau)]$ it follows that (29) holds if and only if $\bar{T}(s;\vartheta;\theta;\rho;\varepsilon,\tau)$ is SPR.
By the Kalman-Yakubovich-Popov Lemma, strict positive realness of $\bar{T}(s;\vartheta;\theta;\rho;\varepsilon,\tau)$ for given $\rho > 0$ is equivalent to the feasibility of the LMI

$$\begin{bmatrix} \mathbf{A}'(\theta;\varepsilon,\tau)\mathbf{X}+\mathbf{X}\mathbf{A}(\theta;\varepsilon,\tau) & \mathbf{X}\mathbf{B}-\mathbf{C}'(\vartheta;\theta;\rho;\varepsilon,\tau) \\ \mathbf{B}'\mathbf{X}-\mathbf{C}(\vartheta;\theta;\rho;\varepsilon,\tau) & -\mathbf{D}(\vartheta;\theta;\rho;\varepsilon,\tau)-\mathbf{D}'(\vartheta;\theta;\rho;\varepsilon,\tau) \end{bmatrix} < 0$$
$$\mathbf{X} = \mathbf{X}' > 0 \quad (30)$$

In turn, by (22),(26),(27), and taking $\mu = \rho^{-1}$, the latter condition can be rewritten as the second and third of (24). Moreover, the first of (24) is equivalent to strict positive realness of $\bar{T}(s;\vartheta;\theta;\rho;\varepsilon,\tau)$ for $\rho = 0$, i.e., strict positive realness of $\Psi(s;\vartheta;\theta;\varepsilon,\tau)$ (see (20)), and hence it holds for all $\vartheta \in \Theta_\theta$. Finally, the set of additional constraints (25) force ϑ to belong to Θ. Therefore, the maximization of $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ over $\Theta_\theta \cap \Theta$ amounts to solving the GEVP (23),(24),(25). ■

Remark 4: From the proof of the above theorem, it is readily checked that for given $\rho > 0$, the solution of the LMI feasibility problem (30) with the additional condition (25) is equivalent to the existence of $\vartheta \in \Theta_\theta \cap \Theta$ such that $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau) > \rho$.

Remark 5: Condition 2 of Lemma 3 implies that for any $\theta \in \Theta_{stab} \cap \Theta$ the value of $\rho_\Phi(\theta;\varepsilon,\tau)$ can be computed by maximizing ρ such that the LMI (30) with $\vartheta = \theta$ holds for some $\mathbf{X} = \mathbf{X}' > 0$. According to Theorem 1, for sufficiently small ε and τ, the solution of this LMI problem also provides an approximation from below of the parametric stability margin $\rho(\theta)$.
It is worth noting that several iterative/heuristic algorithms can be devised to achieve local maximization of $\rho(\vartheta)$ based on the computation of the lower bounds $\rho_\Psi(\vartheta;\theta;\varepsilon,\tau)$ for different values of θ. One possible idea goes as follows: start at some $\theta \in \Theta_{stab} \cap \Theta$, and pick a value $\rho > \rho_\Phi(\theta;\varepsilon,\tau)$. Then, according to Theorem 2 and Remark 4, find, if possible, some value $\vartheta_f \in \Theta$ of ϑ ensuring that $\rho_\Psi(\vartheta_f;\theta;\varepsilon,\tau) > \rho$. If such ϑ_f exists, then by Lemma 3 $\vartheta_f \in \Theta_{stab} \cap \Theta$ and ρ is a lower bound for $\rho_\Phi(\vartheta_f;\varepsilon,\tau)$. If possible, repeat the procedure with θ equal to the value ϑ_f previously found and a larger value of ρ, otherwise try with a smaller ρ, until some optimality criterion is met.
Due to space limitations, we omit the details of the algorithm sketched above. For a comprehensive description and some application examples we refer the reader to [21].

IV. Conclusion

In the present paper the problem of maximizing the real l_2 parametric stability margin via the design of restricted complexity controllers (RCSMM problem) has been addressed. The class of SISO uncertain plants with rank one parametric perturbations and a controller family characterized by transfer functions of fixed degree, which depend affinely on a tunable parameter vector, have been considered.
The main contribution of the paper is the development of a lower bound of the parametric stability margin, which provides a locally optimal approximation in some neighborhood of any given stabilizing controller, and whose maximization is performed via the solution of a Generalized Eigenvalue Problem (GEVP). It is believed that the lower bound can be fruitfully exploited to devise LMI-based algorithms to solve the RCSMM problem. The development of efficient optimization procedures based on the proposed characterization is the subject of current research.

V. Appendix

Proof of Theorem 1: Condition 1. follows from [15] where it is shown (see proof of Theorem 2) that $\Phi(s;\vartheta;\varepsilon,\tau)$ is SPR for sufficiently small ε and τ. More specifically, it is shown that $\Phi(s;\vartheta;0,0)$ is positive real and ε and τ are introduced to ensure that $\Phi(s;\vartheta;\varepsilon,\tau)$ is SPR. In particular, ε takes into account the presence of a singularity at $s = 0$ of $\Pi(s;\vartheta)$ in (8) when q_ϑ is odd, while τ is for making $\Phi(s;\vartheta;\varepsilon,\tau)$ biproper.
Let us now consider condition 2. For any $\vartheta \in \Theta_{stab}$, it can be shown (see Lemma 5, Lemma 6, and eq. (23) in [15]) that

$$\gamma_\Phi(\omega;\vartheta;0,0) = \frac{\bar{R}'(\omega;\vartheta)\bar{I}(\omega;\vartheta)}{\|\bar{I}(\omega;\vartheta)\|_2^2} \quad \forall\omega \notin \Omega_\vartheta.$$

Hence (see Lemma 2 in [15] and (11)), under the assumption $\Omega_\vartheta = \{0\}$, and taking into account that the degree of the closed loop characteristic polynomial is invariant with respect to δ, the parametric stability margin $\rho(\vartheta)$ is given by

$$\rho(\vartheta) = \min\{\rho_0(\vartheta), \hat{\rho}(\vartheta)\}$$

where

$$\rho_0(\vartheta) = \|\bar{R}(0;\vartheta)\|_2^{-1}$$

and

$$\hat{\rho}(\vartheta) = \inf_{\omega>0} r_\Phi(\omega;\vartheta;0,0).$$

Let us consider the expression $\|\bar{R}(\omega;\vartheta) - \gamma\bar{I}(\omega;\vartheta)\|_2$ for $\gamma \in \mathbb{R}$. It reduces to $\|\bar{R}(0;\vartheta)\|_2$ for $\omega = 0$, while for any $\omega > 0$ its minimum with respect to γ is achieved for $\gamma = \gamma_\Phi(\omega;\vartheta;0,0)$. According to (11), this implies

$$r_\Phi(0;\vartheta;\varepsilon,\tau) = \rho_0(\vartheta) \quad \forall \varepsilon \geq 0, \tau \geq 0 \tag{31}$$

and

$$r_\Phi(\omega;\vartheta;\varepsilon,\tau) \leq r_\Phi(\omega;\vartheta;0,0) \quad \forall \omega > 0, \varepsilon > 0, \tau > 0$$

and hence the second inequality in (9) is proven.
To complete the proof, we need to show that for any $\rho < \rho(\vartheta)$ there exist $\varepsilon, \tau > 0$ such that

$$\rho < \inf_{\omega \geq 0} r_\Phi(\omega;\vartheta;\varepsilon,\tau)$$

which, taking (31) into account, reduces to

$$\rho < r_\Phi(\omega;\vartheta;\varepsilon,\tau) \quad \forall \omega > 0. \tag{32}$$

First we note that there exist $\hat{\varepsilon} > 0$ and $\hat{\tau} > 0$ such that $\Phi(s;\vartheta;\varepsilon,\tau)$ is a SPR rational function with relative degree zero for all $0 < \varepsilon \leq \hat{\varepsilon},\ 0 < \tau \leq \hat{\tau}$ and hence $\gamma_\Phi(\omega;\vartheta;\varepsilon,\tau)$ satisfies

$$\lim_{\omega \to 0} \gamma_\Phi(\omega;\vartheta;\varepsilon,\tau) = 0, \quad \lim_{\omega \to +\infty} \gamma_\Phi(\omega;\vartheta;\varepsilon,\tau) = 0$$

for all $0 < \varepsilon \leq \hat{\varepsilon},\ 0 < \tau \leq \hat{\tau}$. Moreover, $\bar{G}(s;\vartheta)$ is strictly proper and hence

$$\lim_{\omega \to +\infty} \bar{R}(\omega;\vartheta) = 0, \quad \lim_{\omega \to +\infty} \bar{I}(\omega;\vartheta) = 0.$$

Therefore, for any given $\rho < \rho(\vartheta)$ there exist $0 < \underline{\omega} < \overline{\omega}$ and arbitrarily small $\bar{\varepsilon} > 0,\ \bar{\tau} > 0$ such that

$$\begin{array}{c} \rho < r_\Phi(\omega;\vartheta;\bar{\varepsilon},\bar{\tau}) \quad \forall \omega \geq \overline{\omega} \\ \rho < r_\Phi(\omega;\vartheta;\bar{\varepsilon},\bar{\tau}) \quad \forall\ 0 < \omega \leq \underline{\omega}, \end{array}$$

where the latter also follows from the fact that

$$\lim_{\omega \to 0} \|\bar{R}(\omega;\vartheta)\|_2 = \|\bar{R}(0;\vartheta)\|_2 = \frac{1}{\rho_0(\vartheta)} \leq \frac{1}{\rho}.$$

Now, it is not difficult to show that $\gamma_\Phi(\omega;\vartheta;\varepsilon,\tau)$ converges uniformly to $\gamma_\Phi(\omega;\vartheta;0,0)$ for $(\varepsilon,\tau) \to (0,0)$ in any compact subset of $\omega > 0$. Then, for any $\rho < \rho(\vartheta)$ there exist $\tilde{\varepsilon}, \tilde{\tau} > 0$ such that, for all $0 \leq \varepsilon \leq \tilde{\varepsilon},\ 0 \leq \tau \leq \tilde{\tau}$,

$$\rho \leq \inf_{\omega \in [\underline{\omega},\overline{\omega}]} r_\Phi(\omega;\vartheta;\varepsilon,\tau) \leq \inf_{\omega \in [\underline{\omega},\overline{\omega}]} r_\Phi(\omega;\vartheta;0,0).$$

Finally, from the observation that $\bar{\varepsilon}$ and $\bar{\tau}$ are arbitrarily small, it follows that $\varepsilon < \min\{\bar{\varepsilon},\hat{\varepsilon},\tilde{\varepsilon}\}$ and $\tau < \min\{\bar{\tau},\hat{\tau},\tilde{\tau}\}$ exist such that (32) holds, thus concluding the proof. ■

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