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Optimal $\mathscr{H}_2$ controllers for spatially invariant systems with delayed communication requirements

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Abstract

We consider the problem of optimal $\mathscr{H}_2$ design of semi-decentralized controllers for a special class of spatially distributed systems. This class includes spatially invariant and distributed discrete-time systems with an inherent temporal delay in the interaction of neighboring sites. We consider the problem of optimal design of distributed controllers that have the same information passing delay structure as the plant. We show how for stable plants, the YJBK parameterization of such stabilizing controllers yields a convex parameterization for this class. We then show how the optimal $\mathscr{H}_2$ problem can be solved.

Keywords: Spatial invariance; $\mathscr{H}_2$ control; Distributed systems

1. Introduction

We consider spatially distributed systems where all signals are functions of discrete spatial and temporal indices, e.g. $\{u_i(t)\}$, where both i and t are integers and we interpret each as the spatial and temporal index, respectively. The theory of such spatio-temporal systems has been worked out in some detail. We consider only spatially distributed systems with the additional property that the dynamics are spatially invariant. For recent work on this class and some of the background for the present work, we refer the reader to [2,4] and the references therein for $\mathscr{H}_2$- and $\mathscr{L}_2$-induced norm minimization problems.

One of the major issues in the design of such distributed controllers is the communications requirements between individual controller sub-systems. One of the applications of this design methodology is to design controllers for large arrays of micro-electro-mechanical system (MEMS), in which there are potentially tens of thousands of actuator/sensor and imbedded control sub-systems. For systems of this size and configuration, centralized controllers are not an option. It turns out that optimally designed centralized controllers have an

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inherent localization property which enables them to be implemented using distributed control elements with limited communication requirements [2,8]. In general, however, the optimal controller does not inherit the structure of the plant and also it is not clear how a prescribed localization rate can be achieved.

Several researchers have recently been looking at the problem of explicitly imposing a priori constraints on communication requirements between controller subsystems. Among these are approaches based on linear matrix inequalities (LMIs) and convex optimization techniques (see [1,3,4] and the references therein) which provide stability and guaranteed $\mathscr{L}_2$ performance levels for systems captured by rational, multidimensional transfer functions. The same structure of controllers as the plant is imposed and a relaxation is used to obtain stability and performance conditions via LMIs.

In this paper, we consider the case where the controller is constrained such that information is propagated at a certain speed between sub-systems, a property we refer to as cone causality. We show that by employing the standard YJBK parametrization approach (e.g., [9,5]), the constraints on the controller transform to convex constraints on the Youla parameter, given that the plant has the same cone-causality structure. We then show how the optimal $\mathscr{H}_2$ problem can be solved based on the input–output approach. By doing so, a suitably relaxed problem is defined and solved that is of interest on its own right, which also provides the means to approximate arbitrarily close the optimal cost via finite-dimensional problems.

2. Preliminaries

We consider signals that are functions of both discrete time t and discrete space i, and are denoted by $u(i,t)$ or equivalently $u_i(t)$. Systems that act on such signals via convolutions are called spatially invariant spatio-temporal systems, and are represented by $y = Gu$, where

$$y_i(t) = \sum_{j=-\infty}^{\infty} \sum_{\tau=-\infty}^{\infty} \hat{g}_{i-j}(t-\tau)u_j(\tau),$$

where the function of two indices $\hat{g}_i(t)$ represents the spatio-temporal impulse response of G. In general, temporal causality is assumed, and it implies that $\hat{g}_i(t)=0$ for $t<0$. It is sometimes helpful to view such systems as a family (indexed by i) of standard temporal systems $\{g_i\}$. Each member of this family has a corresponding λ-transform $g_i(\lambda) = \sum_{t=0}^{\infty} \hat{g}_i(t)\lambda^t$, while the overall spatio-temporal transfer functions is given by $G(z,\lambda) := \sum_{i,t} \hat{g}_i(t)\lambda^t z^i$. This spatio-temporal transfer function can also be expressed as $G(z,\lambda) = \sum_{i=-\infty}^{\infty} g_i(\lambda)z^i$, or alternatively, $G(z,\lambda) = \sum_{t=0}^{\infty} G_t(z)\lambda^i$ where $G_t(z) = \sum_{i=-\infty}^{\infty} \hat{g}_i(t)z^i$.

It is a standard fact that the transform $Y(z,\lambda)$ of the output $y = \{y_i(t)\colon\ -\infty \leqslant i \leqslant \infty, 0 \leqslant t \leqslant \infty\}$ of G to a spatio-temporal input sequence $u = \{u_i(t)\colon\ -\infty \leqslant i \leqslant \infty, 0 \leqslant t \leqslant \infty\}$ with transform $U(z,\lambda)$ relates simply as $Y(z,\lambda)=G(z,\lambda)U(z,\lambda)$ which is the equivalent of the two-dimensional convolution given as $y_i(t)= \sum_{j=-\infty}^{j=\infty} \sum_{\tau=0}^{\tau=t} \hat{g}_{i-j}(t-\tau)u_j(\tau)$.

For the system G, its ℓ_1 norm is $\|G\|_1 := \sum_{i,t} |\hat{g}_i(t)| < \infty$. Note also $\|G\|_1 = \sum_{i=-\infty}^{\infty} \|g_i\|_1$ where $\|g_i\|_1 = \sum_{t=0}^{\infty} |\hat{g}_i(t)|$ is the usual temporal ℓ_1 norm. Similar to the one-dimensional case, the ℓ_2 norm of G can be defined as $\|G\|_2 := (\sum_{i,t} |\hat{g}_i(t)|^2)^{1/2}$ and $\mathscr{H}_2$ norm of its transform $G(z,\lambda)$ given as $\|G\|_{\mathscr{H}_2} := ((1/2\pi)^2 \int_{0\leqslant\theta\leqslant 2\pi} \int_{0\leqslant\omega\leqslant 2\pi} |G(\mathrm{e}^{\mathrm{j}\theta},\mathrm{e}^{\mathrm{j}\omega})|^2 \,\mathrm{d}\omega\,\mathrm{d}\theta)^{1/2}$; the isometry $\|G\|_{\mathscr{H}_2} = \|G\|_2 = \langle G,G\rangle^{1/2}$ also holds. Note that $\|G\|_2 = (\sum_{i=-\infty}^{\infty} \|g_i\|_2^2)^{1/2}$ where $\|g_i\|_2 = (\sum_{t=0}^{\infty} |\hat{g}_i(t)|^2)^{1/2}$ is the usual temporal ℓ_2 norm (which is also equal to the usual $\mathscr{H}_2$ norm of the temporal system g_i). The system G will be called $\mathscr{L}_2$-stable if $\|G\|_{\mathscr{H}_\infty} := \operatorname{ess\,sup}_{0\leqslant\theta<2\pi, |\lambda|\leqslant 1} |G(\mathrm{e}^{\mathrm{j}\theta},\lambda)| < \infty$. This is the kind of stability we consider in this paper, and we refer to such systems simply as stable. To avoid proliferation of notation, the norm subscripts will be dropped as it will be clear from the context.

Given such a stable G we can also view it as a bounded linear mapping of ℓ_2 spatio-temporal input sequences, or, alternatively via the spatio-temporal transforms, a bounded map of $\mathscr{L}_2$ functions. A particular factorization that will be useful for our development is $G = G_{\rm in}G_{\rm out}$ where $G_{\rm in}$ is an isometry on $\mathscr{L}_2$ and $G_{\rm out}$ is (temporally) causally invertible on $\mathscr{L}_2$. It can be easily shown that a particular way to accomplish this factorization is to perform a *family* of standard inner–outer factorizations (e.g., [9,5]) of $G(z,\lambda)$ by viewing it as a temporal system for each (fixed) $z = e^{j\theta}$, i.e., $G(e^{j\theta},\lambda) = G_{\rm in}(e^{j\theta},\lambda)G_{\rm out}(e^{j\theta},\lambda)$. Assuming for technical simplicity that $\inf_{\theta,\omega}|G(e^{j\theta},e^{j\omega})| > 0$, we have that by Parseval's equality, the spatio-temporal system $G_{\rm in}(e^{j\theta},\lambda)$ is an isometry and $G_{\rm out}(e^{j\theta},\lambda)$ is (temporally) causally invertible. As in the temporal case, given a $G(z,\lambda)$ we define $G^{\sim}(z,\lambda) := G(z^{-1},\lambda^{-1})$. The isometry of the inner function $G_{\rm in}(e^{j\theta},\lambda)$ then implies that $G_{\rm in}^{\sim}(e^{j\theta},\lambda)G_{\rm in}(e^{j\theta},\lambda) = 1$. With some abuse of terminology, we call this factorization of a spatio-temporal systems $G = G_{\rm in}G_{\rm out}$ an inner–outer factorization. We stress however, that this is not an inner–outer factorization in the sense used in the mathematical literature on several complex variables, where inner and outer refer to functions whose zero sets are inside and outside the unit polydisk, respectively. The conditions for this latter type of inner–outer factorization are much more stringent, and it is known that such factorizations often do not exist [6].

3. Problem definition

Consider the standard configuration for disturbance attenuation in Fig. 1 where the disturbance w, the regulated output z, the measurements y and the controls u are spatio-temporal sequences, and, the plant G and the controller K are spatially and temporally invariant systems.

The particular structure of interest is when in the spatial and temporal transform domain G_{22}, the map from u to y, is of the form

$$G_{22}(z,\lambda) = \sum_{i=-\infty}^{\infty} g_i(\lambda)z^i \tag{1}$$

with

$$g_i(\lambda) = \lambda^{|i|}\tilde{g}_i(\lambda),$$

where, as presented in the previous section, λ corresponds to the temporal one-sided transform variable, z corresponds to the spatial two-sided transform variable; $\tilde{g}_i(\lambda)$ is a transfer function corresponding to a temporally causal system. The interpretation of this structure is that the input u_i to the ith system g_i affects

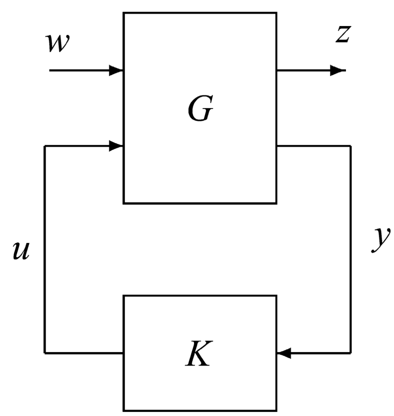

Fig. 1. The standard problem.

the output y_j of the jth system g_j which is $|j-i|$ spatial locations away with a (time) delay proportional to their spatial distance $|j-i|$, i.e., with a delay of $|j-i|$ time steps.

Given such a G_{22}, we are looking for stabilizing controllers with the same structure as G_{22}. Namely, we want K as

$$K(z,\lambda) = \sum_{i=-\infty}^{\infty} k_i(\lambda) z^i \tag{2}$$

with

$$k_i(\lambda) = \lambda^{|i|}\tilde{k}_i(\lambda).$$

Thus, we are imposing an implicit decentralized structure on K since now the measurements of the jth location will be made available at the ith station after $|j-i|$ time steps.

The problem of interest is to design such a K which is stabilizing and minimizes the $\mathscr{H}_2$ norm of the closed loop.

4. Problem solution

We consider here only the case where G_{22} is stable. The following proposition shows that by employing the Youla parameterization the decentralization constraints on K transform to convex constraints on the Youla parameter Q.

Proposition 4.1. *All stabilizing K with the desired structure are given by*

$$K = -Q(I - G_{22}Q)^{-1},$$

with Q stable given by

$$Q(z,\lambda) = \sum_{i=-\infty}^{\infty} q_i(\lambda) z^i$$

and with q_i of the form

$$q_i(\lambda) = \lambda^{|i|}\tilde{q}_i(\lambda),$$

where $\tilde{q}_i$ is stable.

Proof. That all stabilizing, possibly without the structure, K are given as $K = -Q(I - G_{22}Q)^{-1}$ follows by the same arguments as in the finite-dimensional case. In particular, as G_{22} is stable, it always has a trivial (right and left) factorization in the ring of stable systems. Hence, by Theorem 12, p. 364 in [9] all stabilizing K are given by the previous fractional expression.

We will thus prove the structural property of Q only. We view G_{22} as the following mapping:

$$\begin{pmatrix} \vdots \\ y_{-1} \\ y_0 \\ y_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \\ \cdots & g_1 & g_0 & g_{-1} & \cdots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{pmatrix}.$$

Grouping together the outputs at each time step $0, 1, 2, \ldots$ as

$$y(0) = \begin{pmatrix} \vdots \\ y_{-1}(0) \\ y_0(0) \\ y_1(0) \\ \vdots \end{pmatrix}, \qquad y(1) = \begin{pmatrix} \vdots \\ y_{-1}(1) \\ y_0(1) \\ y_1(1) \\ \vdots \end{pmatrix}, \ldots$$

and similarly for u we can view $y = G_{22}u$ as a standard matrix

$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{pmatrix} = \begin{pmatrix} G_0 & & & \\ G_1 & G_0 & & \\ G_2 & G_1 & G_0 & \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \end{pmatrix},$$

where G_i are band operators. In particular, we have

$$G_0 = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \cdots & 0 & \hat{g}_0(0) & 0 & \cdots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

i.e., diagonal,

$$G_1 = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \cdots & 0 & \hat{g}_1(0) & \hat{g}_0(1) & \hat{g}_1(0) & 0 & \cdots \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

i.e., 3-diagonal, and generalizing, G_i is $(2i+1)$-diagonal, where $\hat{g}_i = \{\hat{g}_i(t)\}_{t=0}^{\infty}$ denotes the pulse response of the system associated with $g_i(\lambda)$. Representing similarly K by

$$K = \begin{pmatrix} K_0 & & \\ K_1 & K_0 & \\ & \ddots & \ddots \end{pmatrix}$$

it is easy to verify that K is required to have K_0 diagonal, K_1 3-diagonal, $\ldots, K_i$ $(2i+1)$-diagonal, etc. By considering the parameterization $K = -Q(I - G_{22}Q)^{-1}$ it is now clear that K has the required structure if and only if Q has precisely the same structure, namely,

$$Q = \begin{pmatrix} Q_0 & & \\ Q_1 & Q_0 & \\ & \ddots & \ddots \end{pmatrix}$$

with Q_i $2i+1$-diagonal which completes the proof. $\square$

With the above parameterization, the problem of interest becomes

$$\inf_Q \|H - UQV\|$$

with

$$Q(z,\lambda) = Q_0(z) + Q_1(z)\lambda + \cdots,$$

where

$$Q_0(z) = q_{0,0}, \qquad Q_1(z) = q_{1,-1}z^{-1} + q_{1,0} + q_{1,1}z,$$
$$Q_2(z) = q_{2,-2}z^{-2} + q_{2,-1}z^{-1} + q_{2,0} + q_{2,1}z + q_{22}z^2,$$
$$\cdots$$

with $q_{i,j}$ scalars and H, U, V stable maps that depend only on G. Since in this paper we are considering scalar spatio-temporal transfer functions we further assume that $V = I$. The idea of the approach is not different in the case of matrix transfer functions.

Do an inner–outer for $U(z,\lambda)$ fixing z:

$$U(z,\lambda) = U_{\text{in}}(z,\lambda)U_{\text{out}}(z,\lambda).$$

Let

$$R := U_{\text{out}}Q = R_0(z) + R_1(z)\lambda + \cdots;$$

then

$$\inf_Q \|H - UQ\| = \inf_Q \|U_{\text{in}}^{\sim}H - U_{\text{out}}Q\| = \inf_R \|U_{\text{in}}^{\sim}H - R\|.$$

Let

$$U_{\text{out}}(z,\lambda) = U_0(z) + U_1(z)\lambda + \cdots \ .$$

It thus becomes a problem of characterizing the decentralization constraint on R. Clearly the problem is convex but not finite dimensional.

Relaxing Q to be in ℓ_2 (as opposed to $\mathscr{H}_\infty$), corresponds to a minimum distance problem in ℓ_2 (or $\mathscr{L}_2$) to a closed subspace (since U_{out} is a bounded operator with bounded inverse). By the projection theorem (e.g., [7]), it therefore has always a solution. In the case, where the optimal Q is in ℓ_2 it can always be approximated arbitrarily close with a Q in ℓ_1 (for example, FIR sequences). Thus we consider the problem with $Q \in \ell_2$.

Next we consider a relevant relaxation of the original problem that is of interest on its own right in addition to that it approximates the original problem with arbitrary accuracy. Namely, we require that only the first N coefficients in $Q(\lambda,z) = Q_0(z) + Q_1(z)\lambda + \cdots$ are constrained to correspond to band-operators Q_i with $(2i+1)$-diagonal for $i = 0,1,\ldots,N-1$, where N is arbitrary. This equivalently amounts to relaxing the controller structure in exactly the same manner. The interpretation is that if the spatial distance $|i-j|$ between location i and j is greater or equal than N the time delay in obtaining measurement y_j for the use in control decision u_i at location i is N (i.e., it does not grow indefinitely.) Of course, if $|i-j| < N$ the delay is $|i-j|$ time-steps.

If we denote by μ the optimal value of the original problem and μ_N its relaxation as above it is clear that $\mu_N \leqslant \mu$, i.e., μ_N is a lower bound. Since U_{out} is a bounded operator with bounded inverse, it can be shown that $\lim_{N\to\infty} \mu_N = \mu$ and hence we can approach arbitrarily close the solution to the original problem. For details of this and proofs of convergence, we refer the reader to the appendix.

To keep the exposition clear, let us first look at the one-step delay problem, i.e., $N=1$. The only constraint there is that

$$Q_0(z) = q_{0,0} = \text{scalar independent of } z.$$

The following shows how R is affected.

Proposition 4.2. *With the constraint $Q_0(z) = q_{0,0}$ as above,*

$$R = U_{\text{out}} Q \quad \Leftrightarrow \quad R_0(z) = q_{0,0} U_0(z), \qquad q_{0,0} \textit{ is a scalar.}$$

Proof. ($\Rightarrow$) Obvious.
($\Leftarrow$) Let R be s.t. $R_0(z) = q_{0,0} U_0(z)$, $q_{0,0}$ scalar. Consider

$$Q = q_{0,0} + U_{\text{out}}^{-1}(R - U_{\text{out}} q_{0,0})$$

Note that $R - U_{\text{out}} q_{0,0}$ is of the form $\lambda \tilde{R}(z, \lambda)$ and hence

$$Q(z, 0) = q_{0,0}.$$

Moreover,

$$U_{\text{out}} Q = R. \qquad \square$$

Hence, only $R(z, 0) = R_0(z)$ is affected. If we denote

$$X = U_{\text{in}}^{\sim} H = \cdots X_{-1}(z)\lambda^{-1} + X_0(z) + X_1(z)\lambda + \cdots$$

and if Π_M represents the projection operator on a subspace M of $\mathscr{L}_2$, then by applying a standard projection theorem (e.g., [7, p. 51]) we get that the optimal value of R is given by

$$R^{\text{opt}} = \Pi_{\mathscr{H}_2(S_0)}[X] = \Pi_{S_0}[X_0(z)] + X_1(z)\lambda + X_2(z)\lambda^2 + \cdots,$$

where

$$S_0 = \{r(z) \colon r(z) = q_{0,0} U_0(z) \text{ for some scalar } q_{0,0}\}, \qquad \mathscr{H}_2(S_0) = \left\{ X = \sum_{i=0}^{\infty} X_i(z)\lambda^i \in \mathscr{H}_2 \colon X_0(z) \in S_0 \right\}.$$

To find $\Pi_{S_0}[X_0(z)]$ amounts to finding $q_{0,0}^{\text{opt}}$ such that

$$\langle X_0(z) - q_{0,0}^{\text{opt}} U_0(z), q_{0,0} U_0(z) \rangle = 0 \quad \text{for any } q_{0,0}.$$

Therefore

$$q_{0,0}^{\text{opt}} = \langle X_0(z), U_0(z) \rangle / \langle U_0(z), U_0(z) \rangle.$$

For N-step delay the same albeit more complicated procedure holds. The key again is that only the N-coefficients of R are affected as the following shows.

Proposition 4.3. *With the constraint Q_i corresponding to $2i+1$ diagonal operator for $i = 0, 1, \ldots, N-1$ it holds that*

$$R = U_{\text{out}} Q$$

if and only if

$$\begin{pmatrix} R_0(z) \\ \vdots \\ R_{N-1}(z) \end{pmatrix} = \begin{pmatrix} U_0(z) & & & \\ U_1(z) & U_0(z) & & \\ \vdots & & & \ddots \\ U_{N-1}(z) & \cdots & \cdots & U_0(z) \end{pmatrix} L(z)q, \tag{3}$$

where $L(z)$ *is given as*

$$L(z) := \begin{pmatrix} 1 & & & & \\ 1 & (z^{-1}\ 1\ z) & & & \\ 1 & (z^{-1}\ 1\ z) & (z^{-2}\ z^{-1}\ 1\ z\ z^2) & & \\ \vdots & \vdots & \vdots & \cdots & \\ 1 & (z^{-1}\ 1\ z) & (z^{-2}\ z^{-1}\ 1\ z\ z^2) & \cdots & (z^{-N+1}\ \cdots\ z^{-1}\ 1\ z^1\ \cdots\ z^{N-1}) \end{pmatrix}$$

and the vector q *is of the form*

$$q^{\mathrm{T}} := (q_0^{\mathrm{T}}\ q_1^{\mathrm{T}}\ \cdots\ q_{N-1}^{\mathrm{T}}) \tag{4}$$

with

$$q_i^{\mathrm{T}} = (q_{i,-i}\ \cdots\ q_{i,0}\ \cdots\ q_{i,i}). \tag{5}$$

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Let R be s.t. it satisfies the condition of the above proposition for some scalar vector q. For $i = 0,\ldots,N-1$ define $Q_i(z) := \sum_{j=-i}^{j=i} q_{i,j} z^j$ and consider

$$Q_N(z,\lambda) := \sum_{i=0}^{i=N-1} Q_i(z)\lambda^i.$$

Define

$$Q = Q_N + U_{\mathrm{out}}^{-1}(R - U_{\mathrm{out}}Q_N)$$

Note that $R - U_{\mathrm{out}}Q_N$ is of the form $\lambda^N \tilde{R}(z,\lambda)$ and hence Q is of the required form and moreover,

$$U_{\mathrm{out}}Q = R. \qquad \square$$

According to the parameterization (3) of R, computing the optimal solution in this case amounts to calculating

$$R^{\mathrm{opt}} = \Pi_{\mathscr{H}_2(S_{N-1})}[X] = \chi_0(z) + \chi_1(z)\lambda + \cdots + \chi_{N-1}(z)\lambda^{N-1} + X_N(z))\lambda^N + X_{N+1}(z))\lambda^{N+1} + \cdots,$$

where

$$\begin{pmatrix} \chi_0(z) \\ \chi_1(z) \\ \vdots \\ \chi_{N-1}(z) \end{pmatrix} = \Pi_{S_{N-1}} \begin{pmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{N-1}(z) \end{pmatrix}$$

with

$$S_{N-1}=\left\{r(z)\colon r(z)=\sum_{\substack{i=0,N-1\\ j=-i,i}} q_{i,j}V_{i,j}(z) \text{ for some } q\right\},$$

where the vector of scalars q is of form (4), (5) and

$$V_{i,j}(z)=\begin{pmatrix} \left.\begin{matrix}0\\0\\\vdots\\0\end{matrix}\right\} i \\ U_0(z)\\ U_1(z)+U_0(z)\\ \vdots \\ U_{N-1-i}+\cdots+U_0\end{pmatrix} z^j.$$

Then, by standard projection arguments, the vector q^{opt} can be obtained as the solution of a set of linear equations

$$Aq^{\text{opt}}=p, \tag{6}$$

where the matrix A and the vector p are as

$$A=\begin{pmatrix} \langle V_{0,0},V_{0,0}\rangle & \langle V_{1,-1},V_{0,0}\rangle & \langle V_{1,0},V_{0,0}\rangle & \langle V_{1,1},V_{0,0}\rangle & \langle V_{2,-2},V_{0,0}\rangle & \cdots \\ \langle V_{0,0},V_{1,-1}\rangle & \langle V_{1,-1},V_{1,-1}\rangle & \langle V_{1,0},V_{1,-1}\rangle & \langle V_{1,1},V_{1,-1}\rangle & \langle V_{2,-2},V_{1,-1}\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$p=\begin{pmatrix} \langle (X_0\cdots X_{N-1})^{\mathrm{T}},V_{0,0}\rangle \\ \langle (X_0\cdots X_{N-1})^{\mathrm{T}},V_{1,-1}\rangle \\ \langle (X_0\cdots X_{N-1})^{\mathrm{T}},V_{1,0}\rangle \\ \langle (X_0\cdots X_{N-1})^{\mathrm{T}},V_{1,1}\rangle \\ \vdots \end{pmatrix}.$$

Remarks. As it can be shown (see the appendix), $Q_N^{\text{opt}}(z,\lambda)=Q_0(z)+Q_1(z)\lambda+\cdots$ converges weakly to the (unique) optimal solution Q^{opt} as $N\to\infty$. A sequence of feasible solutions to the original problem can be obtained by considering the truncation

$$Q_N(z,\lambda)=\sum_{i=0}^{N-1} Q_i(z)\lambda^t.$$

Then clearly Q_N satisfies the infinite constraints of the original problem and hence if $v_N := \|H - UQ_N\|$ then $v_N \geqslant \mu$. Moreover, it is immediate that also Q_N converges weakly to the (unique) optimal solution Q^{opt} as $N \to \infty$.

We would also like to emphasize that the relaxed problem is of interest in its own right and not necessarily only due to its connection with the original problem posed. Indeed, several localized controller and plant structures can be studied that need not impose an infinite number of constraints on Q. This is a subject of current research by the authors.

5. Examples

Example 1. Consider the following spatio-temporal system

$$y(i,k+1) - y(i,k) = \frac{T}{L^2}\, y(i+1,k) - 2y(i,k) + y(i-1,k) - \varepsilon y(i,k) + u(i,k).$$

This system comes from the finite-difference discretization of a certain PDE. For our current purpose, however, it serves as an example of a system with nearest neighbor interaction, where effects travel at the speed of one neighbor per time unit. Taking the appropriate transforms one obtains the transfer function

$$G(z,\lambda) = \frac{y(z,\lambda)}{u(z,\lambda)} = \frac{T\lambda}{1 - (\gamma/2)(z^{-1} + 2\alpha + z)\lambda},$$

where

$$\gamma = \frac{2T}{L^2}, \qquad \alpha = \frac{L^2}{(2T)} - \frac{\varepsilon L^2}{2 - 1}.$$

It can be easily checked that the dynamics of such a system are asymptotically stable under the following conditions:

$$\gamma < 1/(1+\alpha), \quad \alpha > 1 - 1/\gamma,$$

which in turn correspond to

$$\varepsilon > 0, \quad T < \frac{1}{2/L^2} + \varepsilon/2.$$

Moreover, $G(z,\lambda)$ is of form (1),

$$G(z,\lambda) = \sum_{i=1}^{\infty} T\left(\frac{\gamma}{2}\right)^{i-1}(z^{-1} + 2\alpha + z)^{i-1}\lambda^i.$$

We want to compute a decentralized controller for optimal $\mathscr{H}_2$ attenuation of an additive disturbance on the system output with weighting function

$$W(z,\lambda) = \frac{\lambda}{1 - (c/2)(z^{-1} + 2a + z)\lambda}$$

which is of the same structure as the plant itself. We assume $W(z,\lambda)$ to be asymptotically stable as well, i.e.,

$$c < 1/(1+a), \quad a > 1 - 1/c.$$

With the stabilizing controller parameterization

$$K(z,\lambda) = \frac{-Q(z,\lambda)}{1 - G(z,\lambda)Q(z,\lambda)}$$

with $K(z,\lambda)$ and $Q(z,\lambda)$ of the prescribed form, the problem can be stated as

$$\min_Q \|(1 - GQ)W\| = \min_Q \|H - UQ\|,$$

where

$$H(z,\lambda)=\frac{\lambda}{1-r(z)\lambda},\qquad U(z,\lambda)=\frac{T\lambda^2}{(1-\rho(z)\lambda)(1-r(z)\lambda)}$$

and

$$\rho(z)=\frac{\gamma}{2}(z^{-1}+2\alpha+z),\qquad r(z)=\frac{c}{2}(z^{-1}+2a+z).$$

An inner–outer factorization of $U(z,\lambda)$ yields

$$U_{\mathrm{in}}(z,\lambda)=\lambda^2,\qquad U_{\mathrm{out}}(z,\lambda)=\frac{T}{(1-\rho(z)\lambda)(1-r(z)\lambda)}.$$

Hence

$$\begin{aligned}X(z,\lambda)&=U_{\mathrm{in}}^{\sim}(z,\lambda)H(z,\lambda)=\frac{1}{\lambda(1-r(z)\lambda)}\\&=X_{-1}(z)\lambda^{-1}+X_0(z)+X_1(z)\lambda+X_2(z)\lambda^2+\cdots\\&=\lambda^{-1}+r(z)+r^2(z)\lambda+r^3(z)\lambda^2+\cdots\end{aligned}\tag{7}$$

and

$$\begin{aligned}U_{\mathrm{out}}(z,\lambda)&=U_0(z)+U_1(z)\lambda+U_2(z)\lambda^2+\cdots\\&=T+T(r(z)+\rho(z))\lambda+T(r^2(z)+r(z)\rho(z)+\rho^2(z))\lambda^2+\cdots,\end{aligned}$$

i.e.,

$$U_i(z)=T\sum_{j=0}^{i}r^{i-j}(z)\rho^j(z).$$

Let us compute the solution to the $N=2$ relaxed problem. We get

$$V_{0,0}(z)=T[1\quad 1+r(z)+\rho(z)]^{\mathrm{T}},\qquad V_{1,i}(z)=T[0\quad 1]^{\mathrm{T}}z^i,\qquad i=-1,0,1.$$

Hence, by computing the inner products,

$$A=T^2\begin{pmatrix}2(1+ac+\alpha\gamma)+2\left(\frac{\gamma}{2}+\frac{c}{2}\right)^2+(\alpha\gamma+ac)^2 & \frac{\gamma}{2}+\frac{c}{2} & 1+\alpha\gamma+ac & \frac{\gamma}{2}+\frac{c}{2}\\ \frac{\gamma}{2}+\frac{c}{2} & 1 & 0 & 0\\ 1+\alpha\gamma+ac & 0 & 1 & 0\\ \frac{\gamma}{2}+\frac{c}{2} & 0 & 0 & 1\end{pmatrix},$$

$$p=T\begin{pmatrix}ac+\frac{c^2}{2}(1+2a^2)+\frac{ac^3}{2}(2a^2+3)+\frac{c^2\gamma}{2}(2a+\alpha+2a^2\alpha)\\ ac^2\\ c^2(a^2+1/2)\\ ac^2\end{pmatrix}.$$

Solving (6) we have

$$q^{\rm opt} = \begin{pmatrix} q^{\rm opt}_{0,0} \\ q^{\rm opt}_{1,-1} \\ q^{\rm opt}_{1,0} \\ q^{\rm opt}_{1,1} \end{pmatrix} = T^{-1} \begin{pmatrix} ac \\ ac^2/2 - ac\gamma/2 \\ c^2/2 - \alpha\gamma ac - ac \\ ac^2/2 - ac\gamma/2 \end{pmatrix}.$$

The optimal truncated solution to the two-step relaxation is then given by

$$Q_2(z,\lambda) = T^{-1}ac - T^{-1}[(ac\gamma/2 - ac^2/2)z^{-1} + (\alpha\gamma ac - c^2/2 + ac) + (ac\gamma/2 - ac^2/2)z]\lambda$$

We note that in this case $U_{\rm out}(z,\lambda)$ satisfies the cone causality property and therefore $R(z,\lambda) = U_{\rm out}(z,\lambda)Q(z,\lambda)$ preserves such property. Hence, the value of $R(z,\lambda)$ corresponding to the optimal solution (not only to the N-relaxed case but also to the original fully decentralized problem) is obtained by keeping only the terms of the (temporally) causal part of $X(z,\lambda)$ which form the cone structure. Therefore, from (7),

$$R^{\rm opt}(z,\lambda) = ac + (ac^2z^{-1} + c^2(a^2 + 1/2) + ac^2z)\lambda + \cdots.$$

The optimal solution to the $\mathscr{H}_2$ problem without decentralization constraints is given by

$$Q^{\rm o} = U^{-1}_{\rm out}\Pi_{\mathscr{H}_2}[HU^{-1}_{\rm in}] = T^{-1}(1 - r(z)\lambda)(1 - \rho(z)\lambda)\left(\frac{r(z)}{1 - r(z)\lambda}\right)$$

$$= T^{-1}r(z)(1 - \rho(z)\lambda) = T^{-1}\frac{c}{2}(z^{-1} + 2a + z) - T^{-1}\frac{c\gamma}{4}(z^{-1} + 2a + z)(z^{-1} + 2\alpha + z)\lambda,$$

which does not have the sought cone structure.

Assuming the following numerical values:

$$\alpha = 1, \quad \gamma = \tfrac{1}{3}, \quad a = 1, \quad c = \tfrac{1}{4}$$

and computing the value v_2 of the cost functional associated to the truncated 2-relaxed solution yields

$$v_2 = \|H(z,\lambda) - U(z,\lambda)Q_2(z,\lambda)\| = 1.0659.$$

For the optimal decentralized solution to the non-relaxed problem we have

$$X(z,\lambda) - R^{\rm opt}(z,\lambda) = \lambda^{-1} + \sum_{i=1}^{\infty}\left(\frac{c}{2}\right)^i (z^{-i} + z^i)\lambda^{i-1} = \lambda^{-1}\left(\frac{1}{1 - (c/2)z^{-1}\lambda} + \frac{1}{1 - (c/2)z\lambda} - 1\right)$$

yielding a cost functional value

$$\mu = \|X - R^{\rm opt}\| = 1.0157.$$

It is easily shown that the cost of the optimal centralized $\mathscr{H}_2$ solution is given by

$$\|H(z,\lambda) - U(z,\lambda)Q^{\rm o}(z,\lambda)\| = 1.$$

It should be noted that the truncated solution to the $N = 2$ that corresponds to v_2 is within 6% of the optimal solution μ.

Example 2. In the previous example, assume the disturbance weighting function $W(z,\lambda)$ to be

$$W(z,\lambda) = \frac{w(z)\lambda}{1 - r(z)\lambda},$$

where

$$w(z) = \frac{d}{2}(z^{-1} + 2e + z).$$

We get

$$H(z,\lambda)=\frac{w(z)\lambda}{1-r(z)\lambda},\qquad U(z,\lambda)=\frac{Tw(z)\lambda^2}{(1-\rho(z)\lambda)(1-r(z)\lambda)}.$$

Factorizing $U(z,\lambda)$ yields

$$U_{\rm in}(z,\lambda)=\lambda^2,\qquad U_{\rm out}(z,\lambda)=\frac{Tw(z)}{(1-\rho(z)\lambda)(1-r(z)\lambda)}$$

and hence

$$X(z,\lambda)=U_{\rm in}^{\sim}(z,\lambda)H(z,\lambda)=\frac{w(z)}{\lambda(1-r(z)\lambda)}$$

$$=w(z)\lambda^{-1}+w(z)r(z)+w(z)r^2(z)\lambda+w(z)r^3(z)\lambda^2+\cdots$$

and

$$U_{\rm out}(z,\lambda)=Tw(z)+Tw(z)(r(z)+\rho(z))\lambda+Tw(z)(r^2(z)+r(z)\rho(z)+\rho^2(z))\lambda^2+\cdots$$

Note that in this case $U_{\rm out}(z,\lambda)$ misses the cone structure. Hence, we cannot compute the fully decentralized solution explicitly as in the previous example and we need to look at the relaxed problem. Again, let $N=2$. We get

$$V_{0,0}(z)=T[w(z)\quad w(z)(1+r(z)+\rho(z))]^{\rm T},\qquad V_{1,i}(z)=T[0\quad w(z)]^{\rm T}z^i,\qquad i=-1,0,1.$$

Assuming

$$\alpha=1,\qquad \gamma=\tfrac{1}{3},\qquad a=e=2,\qquad c=d=\tfrac{1}{10}$$

computing the inner products and solving for $q^{\rm opt}$ yields the following feasible solution Q_2

$$Q_2(z,\lambda)=0.2445T^{-1}+T^{-1}[-0.0315z^{-1}-0.3309-0.0315z]\lambda$$

with cost

$$v_2=0.2247.$$

Again, the optimal centralized solution is given by

$$Q^{\rm o}=T^{-1}r(z)(1-\rho(z)\lambda)=T^{-1}\frac{c}{2}(z^{-1}+2a+z)-T^{-1}\frac{c\gamma}{4}(z^{-1}+2a+z)(z^{-1}+2\alpha+z)\lambda$$

with cost

$$\|H(z,\lambda)-U(z,\lambda)Q^{\rm o}(z,\lambda)\|=\|w(z)\lambda\|=\frac{d}{2}\sqrt{2+4{\rm e}^2}=0.2121.$$

Note that $Q^{\rm o}$ again does not have the structure. Also, v_2 is at most within 6% of optimal.

6. Conclusion

We considered optimal $\mathscr{H}_2$ control for stable distributed discrete-time systems with an inherent temporal delay in the interaction of neighboring sites. When the same information passing delay structure is posed on the controller, thus imposing a decentralization constraint, it was shown that the problem is a convex one by employing the YJBK parameterization. A method for obtaining an exact solution was given for finite delay in the transmission of information from site to site in the distributed controller.

The case of unstable systems can be similarly treated under appropriate assumptions. Also, the same input–output approach is useful to provide computationally tractable design methods when other performance criteria are of interest. These topics are the subject of current research by the authors.

Appendix

Lower bound convergence. Let

$$\mu_N := \|H - UQ_N^{\text{opt}}\| \quad = \|X - U_{\text{out}}Q_N^{\text{opt}}\| = \|X - R_N^{\text{opt}}\|.$$

Obviously,

$$\mu_N \leqslant \mu_{N+1} \leqslant \cdots \leqslant \mu.$$

Claim.

$$\|R_N^{\text{opt}}\| \leqslant 2\|X\| \quad \textit{for all } N.$$

Proof.

$$\|X - R_N\| \geqslant \|R_N\| - \|X\|.$$

If

$$\|R_N\| > 2\|X\|,$$

then

$$\|X - R_N\| > \|X\|.$$

But

$$\|X\| \geqslant \mu_N$$

for any N since $R_N = U_{\text{out}}Q_N = 0$ (obtained with $Q_N = 0$ which has the structure) is a legitimate R_N. Therefore, if

$$\|R_N\| > 2\|X\|,$$

then

$$\|X - R_N\| > \mu_N$$

and so for R_N^{opt} to be optimal it is necessary that

$$\|R_N^{\text{opt}}\| \leqslant 2\|X\|,$$

which completes the proof of the claim. □

Having $\|R_N^{\text{opt}}\|$ bounded uniformly implies $\exists\{R_{N_n}^{\text{opt}}\}$ convergent weakly to $\bar{R}$. Now $\bar{R}$ has the appropriate structure for otherwise contradicts the weak convergence of $\{R_{N_n}^{\text{opt}}\}$. Indeed, suppose that $\bar{R}$ of the form $\bar{R} = U_{\text{out}}\bar{Q}$ with $\bar{Q}$ not having the cone causality. Note that requiring a Q to have cone causality means that

$$\langle Q, F_{tj}\rangle = 0,$$

where

$$F_{tj}(z,\lambda) = z^j\lambda^t, \quad t = 0, 1, \ldots \quad \text{and}$$

$$j = \cdots, -(t+2), -(t+1), (t+1), (t+2), \cdots$$

As $R_{N_n}^{\text{opt}}$ converges weakly to $\bar{R}$ we have that for all $F \in \mathscr{L}_2$

$$\langle R_{N_n}^{\text{opt}} - \bar{R}, F\rangle \to 0, \quad \langle U_{\text{out}}(Q_{N_n}^{\text{opt}} - \bar{Q}), F\rangle \to 0, \quad \langle Q_{N_n}^{\text{opt}} - \bar{Q}, U_{\text{out}}^{*}F\rangle \to 0,$$

where $U_{\rm out}^*$ is the adjoint of $U_{\rm out}$ in $\mathscr{L}_2$. Take $F = (U_{\rm out}^*)^{-1} F_{tj}$ then $F \in \mathscr{L}_2$ as $U_{\rm out}$. $U_{\rm out}^{-1}$ are bounded on $\mathscr{L}_2$. Then

$$\langle Q_{N_n}^{\rm opt} - \bar{Q}, F_{ts} \rangle \to 0,$$

but $\langle Q_{N_n}^{\rm opt}, F_{tj} \rangle = 0$ for sufficiently large N_n. Thus, $\bar{Q}$ has to have the cone structure and hence $\bar{R}$ is generated by such $\bar{Q}$. The rest shows that $\mu_{N_n} \to \mu$ and in fact existence of an optimal $\bar{R}$. By semicontinuity

$$\liminf \|X - R_{N_n}^{\rm opt}\| \geqslant \|X - \bar{R}\|.$$

But

$$\mu \geqslant \|X - R_{N_n}^{\rm opt}\| = \mu_{N_n}.$$

Thus

$$\mu \geqslant \|X - \bar{R}\|,$$

which shows that $\bar{R}$ is the optimal $R^{\rm opt}$ and so is $\bar{Q} = U_{\rm out}^{-1} \bar{R} =: Q^{\rm opt}$ with cone causality satisfied, and, $\mu_{N_n} \to \mu$. It also follows as by projection $R^{\rm opt}$ is unique, that the whole sequence $Q_N^{\rm opt} \to Q^{\rm opt}$ weakly and $\mu_N \to \mu$.

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