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A CHARACTERIZATION OF ROBUST SPR SYNTHESIS FOR SYSTEMS WITH l_p PARAMETRIC UNCERTAINTY

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Abstract. The filter synthesis problem for robust strict positive realness ($RSPR$) of systems with l_p parametric uncertainty is addressed. A general characterization of the solutions of the $RSPR$ problem in this framework is derived. The proposed characterization is then exploited in order to devise synthesis procedures which yield polynomial or rational filters of bounded degree with guaranteed SPR robustness margin in the l_2, l_∞, and l_1 cases.
Keywords. Filter design, robustness, strict positive realness, uncertain systems, parametric uncertainty.

1 Introduction

The invariance of the Strict Positive Realness (SPR) property of rational functions with respect to numerator and denominator perturbations is relevant to many problems in the analysis of absolute stability of nonlinear Lur'e systems and the design of adaptive schemes (see, e.g., [1]-[14]). For instance, convergence of several recursive identification algorithms or adaptive schemes is ensured provided that a suitable family of rational functions enjoys the SPR property (see, e.g., [15]-[17]).

The key issue is the robust SPR ($RSPR$) problem. Given a set $\mathcal{P}$ of polynomials and a region Λ of the complex plane, determine if there exists a polynomial or a rational filter F such that each rational function P/F, $P \in \mathcal{P}$, is strictly positive real over Λ. For instance, in the context of recursive identification schemes, the set $\mathcal{P}$ can be viewed as a model of the uncertainty about the true plant and Λ is the region of the complex plane where the power spectral density of the regressor is concentrated.

Several results are available on the existence and construction of F for different choices of $\mathcal{P}$ and Λ. In [4],[6],[7],[13],[14] the continuous-time and discrete-time robust SPR problems are considered when $\mathcal{P}$ is a polyhedron (l_∞ set) in the coefficient space, while in [8]-[10] the set $\mathcal{P}$ is described in terms of root location regions and Λ is some subset of the complement of the unit disk. A solution to the continuous-time $RSPR$ problem for ellipsoidal (l_2) uncertainty is given in [18], and its discrete-time extension is dealt with in [19]. An LMI characterization of polynomial $RSPR$ filters for l_∞ uncertainty is proposed in [20].

In this paper, we consider the $RSPR$ problem in the unified framework of a set $\mathcal{P}$ given by an l_p ball in coefficient space. Exploiting the results in [6], it is first shown that the stability of the polynomials of $\mathcal{P}$ is a necessary and sufficient condition for the existence of the sought filter F. Then, an analysis based on an important result in [21] is performed, in order to provide a complete characterization of the filters solving the $RSPR$ problem.

In the l_2 case, the proposed characterization leads to the synthesis procedure in [18], which provides a rational solution whose degree is bounded by that of the uncertain polynomial.

The cases of l_∞ and l_1 polynomial families are also investigated. The general characterization is exploited in order to devise a numerical procedure to compute a polynomial filter F with guaranteed robustness margin. Moreover, it is shown that when only the even (odd) coefficients of the polynomial are uncertain, a solution of the $RSPR$ problem is provided by a polynomial filter which can be computed in closed form.

The paper is organized as follows. Section 2 contains the problem formulation and some preliminary results. Section 3 presents a general result characterizing the filters solving the $RSPR$ problem. Section 4 recalls the synthesis results for the l_2 case. Section 5 considers the l_∞ case and Section 6 discusses the l_1 case. Section 7 reports some concluding comments.

Notation.

$\mathbb{C}$: complex plane;
$s \in \mathbb{C}$: complex number;
$\mathrm{Re}[s], \mathrm{Im}[s]$: real and imaginary parts of s;
$\arg[s]$: argument of s;
$P(s)$: real polynomial;
∂P : degree of $P(s)$;
$[P(s)]_o$: polynomial containing only the odd powers of $P(s)$;
$\mathbb{R}^n$: real n-space;
$v = (v_1, \ldots, v_n)'$: vector of $\mathbb{R}^n$ ($'$ denotes transpose);
$\|v\|_p$: p-norm of v;
$\mathcal{H}$: set of Hurwitz polynomials;
$\mathcal{RH}_\infty$: set of stable proper real rational functions.

Basic definitions.

Definition 1 A rational function $\Phi(s)$ is said to be strictly positive real if

1. $\Phi(s), \Phi^{-1}(s) \in \mathcal{RH}_\infty$;
2. $\mathrm{Re}\,[\Phi(j\omega)] > 0 \quad \forall \omega \geq 0$.

Definition 2 For any norm $\|\cdot\|_p$ on $\mathbb{R}^n$, the dual norm $\|\cdot\|_p^d$ is defined as

$$\|x\|_p^d = \max\{x'y : \ \|y\|_p \leq 1\}.$$

In particular, we recall that $\|\cdot\|_2^d = \|\cdot\|_2, \ \|\cdot\|_\infty^d = \|\cdot\|_1, \ \|\cdot\|_1^d = \|\cdot\|_\infty$.

2 Problem formulation

The robust SPR problem in the continuous-time case can be stated as follows [4],[6]. Given a set of polynomials $\mathcal{P}$, determine, if it exists, a polynomial (or in general a rational function) $F(s)$ such that for any polynomial $P(s) \in \mathcal{P}$ the rational function $P(s)/F(s)$ is strictly positive real over the closed right half plane.
In this paper, we address the robust SPR problem for a set of polynomials described by an l_p ball in the coefficient space, centered at some given nominal polynomial $P_0(s)$.

Definition 3 An l_p set of polynomials of degree m is defined as

$$\mathcal{P}_\rho^p := \left\{ P(s) = P_0(s) + \sum_{i=1}^{n} q_i P_i(s) : \|q\|_p \leq \rho \right\}$$

where $P_0(s), \ P_1(s), \ \ldots, P_n(s)$ are such that $\partial P_0 = m$, $\partial P_i < m$ for all $i = 1, \ldots, n$, $q = (q_1 \ldots q_n)' \in \mathbb{R}^n$ is the parameter vector, and $\rho > 0$.

The robust SPR ($RSPR$) problem is stated next.

$RSPR$ **problem.** Given the set $\mathcal{P}_\rho^p$, determine a polynomial or rational function $F(s)$, if it exists, such that the SPR conditions

1. $$\frac{P(s)}{F(s)}, \frac{F(s)}{P(s)} \in \mathcal{RH}_\infty$$
2. $$\mathrm{Re}\left[\frac{P(j\omega)}{F(j\omega)}\right] > 0 \ \ \forall \omega \geq 0. \tag{1}$$

hold for all $P(s) \in \mathcal{P}_\rho^p$.

Since the numerator of a strictly positive real rational rational function is Hurwitz, hereafter without loss of generality we assume that the polynomial $P_0(s)$ is Hurwitz.

Let ρ^* denote the l_p parametric stability margin of $\mathcal{P}^p_\rho$, i.e., the maximal ρ such that $\mathcal{P}^p_\rho$ cointains only Hurwitz polynomials [3].

$$\rho^* = \sup_{\mathcal{P}^p_\rho \subset \mathcal{H}} \rho.$$

It is obvious that the condition $\rho < \rho^*$ is a necessary one for the $RSPR$ problem to have a solution. In the important paper [6], it was shown that such condition is indeed also sufficient when the uncertain polynomial family is described by a polyhedron in coefficient space (l_∞ case). Since the key element of the proof is the convexity of the polynomial set, the result can be proven to hold in the general l_p case, too.

Theorem 1 *Consider the set $\mathcal{P}^p_\rho$ of uncertain polynomials and suppose that $\rho < \rho^*$. Then, there exist a nonnegative integer M and a Hurwitz polynomial $R(s)$ of degree $m + M$ such that the rational function*

$$F(s) = \frac{R(s)}{(s+1)^M} \tag{2}$$

solves the $RSPR$ problem.

Proof. The result parallels that of Theorem 3.1 in [6], once the finite set $\{n_i(s)\}$ is replaced by the convex set $\mathcal{P}^p_\rho$. Indeed, let

$$\overline{\phi}(\omega) =: \sup_{P \in \mathcal{P}^p_\rho} \arg[P(j\omega)] \quad ; \quad \underline{\phi}(\omega) =: \inf_{P \in \mathcal{P}^p_\rho} \arg[P(j\omega)].$$

Since $\mathcal{P}^p_\rho$ is a convex degree-invariant set of Hurwitz polynomials, the functions $\underline{\phi}(\omega)$ and $\overline{\phi}(\omega)$ are well defined and the following condition is true (see [3])

$$\overline{\phi}(\omega) - \underline{\phi}(\omega) < \pi \quad \forall \omega \geq 0. \tag{3}$$

Now, introducing the function

$$\phi^*(\omega) := \frac{\overline{\phi}(\omega) + \underline{\phi}(\omega)}{2},$$

one has that the relation

$$|\arg[P(j\omega)] - \phi^*(\omega)| < \frac{\pi}{2} \quad \forall \omega \geq 0 \tag{4}$$

holds for each polynomial $P(s) \in \mathcal{P}_\rho$.
Thus, for the $RSPR$ problem to be solved, it is enough to show existence of a function $F^*(s)$ such that ${F^*}^{-1}(s) \in \mathcal{RH}_\infty$ and

$$\arg[F^*(j\omega)] = \phi^*(\omega).$$

Employing a series expansion as in [6], it can be shown that $F^*(s)$ can be arbitrarily approximated via a rational function of the form (2) for suitable

$R(s)$ and M. ♢

Although quite interesting from a conceptual viewpoint, Theorem 1 does not provide an efficient design method. Indeed, since $F(s)$ is computed via a procedure based on a series expansion [6], there is no a-priori knowledge of the degree M of the filter $F(s)$.

3 A characterization of the filters solving the $RSPR$ problem

In this section we introduce a new characterization of the $RSPR$ problem which yields efficient procedures for the synthesis of filters F with a-priori bounded degree.
Let

$$G(s) := \left(-\frac{P_1(s)}{P_0(s)}, \ldots, -\frac{P_n(s)}{P_0(s)}\right)' \tag{5}$$

and introduce the two vector functions

$$\begin{aligned} R(\omega) &:= \mathrm{Re}[G(j\omega)], \\ I(\omega) &:= \mathrm{Im}[G(j\omega)]. \end{aligned} \tag{6}$$

It can be checked that the $RSPR$ problem amounts to computing a function $\Phi(s)$ such that

$$\Phi(s), \Phi^{-1}(s) \in \mathcal{RH}_\infty \tag{7}$$

and

$$\mathrm{Re}\left[\Phi(j\omega)\left(1 - q'G(j\omega)\right)\right] > 0 \quad \forall\omega \geq 0 \quad \forall q : \|q\|_p \leq \rho. \tag{8}$$

Then, the filter $F(s)$ is readily obtained via the relation

$$F(s) = \frac{P_0(s)}{\Phi(s)}.$$

The next result provides the sought characterization of the solutions of the $RSPR$ problem.

Theorem 2 *All the rational filters solving the $RSPR$ problem have the form*

$$F(s) = \frac{P_0(s)}{\Phi(s)}$$

where $\Phi(s)$ is a rational function such that

1. *$\Phi(s)$ is strictly positive real;*
2. $$\|R(\omega) - \gamma_\Phi(\omega) I(\omega)\|_p^d < \frac{1}{\rho} \quad \forall\omega \geq 0 \tag{9}$$

being

$$\gamma_\Phi(\omega) := \frac{\text{Im}[\Phi(j\omega)]}{\text{Re}[\Phi(j\omega)]}. \tag{10}$$

Proof. According to Definition 1, condition (7) and inequality (8) for $q = 0$ are equivalent to condition 1. For condition 2, observe first that (8) can be rewritten as

$$\begin{array}{ll} (a) & \text{Re}[\Phi(j\omega)] > 0 \\ (b) & q'\left[R(\omega) - \gamma_\Phi(\omega)I(\omega)\right] < 1 \end{array} \quad \forall\omega \geq 0 \quad \forall q : \|q\|_p \leq \rho. \tag{11}$$

Then, according to Definition 2, the duality property of norms implies that (11(b)) holds for all q such that $\|q\|_p \leq \rho$ if and only if condition 2 holds. $\diamondsuit$

Remark 1 The above theorem is based on an important result in [21]. The main difference is that here we consider the tangent of the argument of $\Phi(j\omega)$ (i.e., $\gamma_\Phi(\omega)$) in place of the argument itself in [21] (see proof of Theorem 1, part 2).

By Theorem 2, a central issue in the solution of the $RSPR$ problem turns out to be the characterization of the set of functions

$$\Gamma_\rho^p = \left\{\gamma(\omega) \ : \ \|R(\omega) - \gamma(\omega)I(\omega)\|_p^d < \frac{1}{\rho} \quad \forall\omega \geq 0\right\}. \tag{12}$$

In particular, a solution exists if and only if there exists a strictly positive real rational function $\Phi(s)$ such that $\gamma_\Phi(\omega) \in \Gamma_\rho^p$.
In the next sections, the set (12) and the existence of $\Phi(s)$ will be investigated for ellipsoidal (l_2), polytopic (l_∞), and l_1 polynomial families. At this stage, we only introduce the general underlying idea for solving the $RSPR$ problem.
Assume that the necessary condition $\rho < \rho^*$ holds, which, according to Theorem 1, implies that Γ_ρ^p is not empty, and introduce the set of frequencies

$$\Omega_0 = \{\omega \geq 0 \quad : \quad I(\omega) = 0\}.$$

We note that for all $\omega \notin \Omega_0$, the inequality in (12) simply provides a constraint on $\gamma(\omega)$ of the form

$$\underline{\gamma}(\omega) < \gamma(\omega) < \overline{\gamma}(\omega) \tag{13}$$

where $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ are suitable functions which depend on ρ. For $\omega \in \Omega_0$, it can be easily seen that the inequality in (12) is implied by condition $\rho < \rho^*$ alone.
For each $\omega \notin \Omega_0$, the band defined by (13) narrows as ρ increases. Consider Fig. 1, in which we assume $\Omega_0 = \{0\}$. In Fig. 1(a), the functions $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ are depicted for a given $\rho = \rho_1 < \rho^*$. Any solution of the $RSPR$ problem is given by a strictly positive real $\Phi(s)$ such that $\gamma_\Phi(\omega)$ belongs to

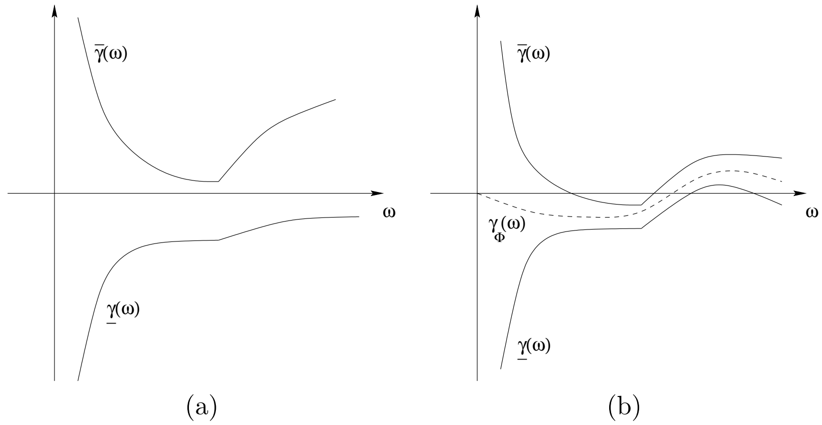


Figure 1: (a): $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for $\rho = \rho_1 < \rho^*$; (b): $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for $\rho = \rho_2$ $(\rho_1 < \rho_2 < \rho^*)$.

the band defined by $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$. In this case, it is easily verified that $\Phi(s) = 1$ solves the $\overline{R}SPR$ problem, since $\gamma_\Phi(\omega) = 0$ is within the band. According to Theorem 2, such a solution leads to the filter $F(s) = P_0(s)$, which is the nominal polynomial itself. It is clear that such a filter is likely to perform well for small uncertainty, i.e., for values of ρ sufficiently smaller than ρ^*. For larger values of ρ, this is no longer guaranteed as shown in Fig. 1(b), where $\rho = \rho_2 > \rho_1$ is considered. In this case a different solution must be found (see dashed line in Fig. 1(b)).
Now, introduce the function

$$\gamma^*(\omega) = \arg\min_{\gamma} \|R(\omega) - \gamma I(\omega)\|_p^d, \quad \omega \notin \Omega_0 \tag{14}$$

which is at each ω the value of $\gamma(\omega)$ minimizing the left hand side of the inequality in (12).
Clearly, the inequality in (12) holds for $\gamma^*(\omega)$ for all $\omega \notin \Omega_0$. This fact suggests the following procedure for obtaining a solution of the $RSPR$ problem: look for a strictly positive real rational function $\Phi(s)$ such that $\gamma_\Phi(\omega)$ is as close as possible to $\gamma^*(\omega)$ for all $\omega \notin \Omega_0$. This general idea is exploited in the next sections in order to devise synthesis methods for l_2, l_∞, and l_1 uncertainty.

4 l_2 uncertainty case

Assume $p = 2$ and suppose $\rho < \rho^*$. According to (12), we need to characterize the set Γ_ρ^2, i.e., the set

$$\Gamma_\rho^2 = \left\{ \gamma(\omega) \ : \ \|R(\omega) - \gamma(\omega) I(\omega)\|_2^2 < \frac{1}{\rho^2} \quad \forall \omega \geq 0 \right\} \tag{15}$$

We have the following result [18].

Proposition 1 *Let $\rho < \rho^*$. Then, the following statements hold.*

1. *Γ_ρ^2 is the set of $\gamma(\omega)$ such that*

$$\underline{\gamma}(\omega) < \gamma(\omega) < \overline{\gamma}(\omega) \quad \forall \omega \notin \Omega_0 \tag{16}$$

where

$$\underline{\gamma}(\omega) = \gamma^*(\omega) - \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}, \quad \overline{\gamma}(\omega) = \gamma^*(\omega) + \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2} \tag{17}$$

being

$$\gamma^*(\omega) = \frac{R'(\omega)I(\omega)}{\|I(\omega)\|_2^2} \tag{18}$$

and

$$\Delta(\omega) = \left[R'(\omega)I(\omega)\right]^2 - \|I(\omega)\|_2^2 \left[\|R(\omega)\|_2^2 - \frac{1}{\rho^2}\right]. \tag{19}$$

2. *Γ_ρ^2 is nonempty.*

The characterization of $\gamma^*(\omega)$ in (18) is exploited in [18] in order to derive the closed-form expression of a rational filter $F(s)$ solving the $RSPR$ problem for all $\rho < \rho^*$. The basic idea is to derive a positive real rational function $\Phi^*(s)$ such that $\gamma_{\Phi^*}(\omega) = \gamma^*(\omega)$ almost everywhere in $\omega \geq 0$, i.e., for all $\omega \notin \Omega_0$, and then to apply a slight perturbation to $\Phi^*(s)$ in order to obtain an SPR rational function $\Phi(s)$ such that

$$\underline{\gamma}(\omega) < \gamma_\Phi(\omega) < \overline{\gamma}(\omega) \quad \forall \omega \notin \Omega_0.$$

Although the proposed approach can provide a solution of the $RSPR$ problem under the only condition $\rho < \rho^*$, in the sequel we will report the related main results under a simplifying assumption concerning the vector function $I(\omega)$ in (6), which can be shown to be violated only in non-generic cases. We refer the reader to [18] for a complete discussion.

Assumption 1. Let $I(\omega) \neq 0$ for all $\omega > 0$, i.e., $\Omega_0 = \{0\}$.

Let us introduce the polynomial

$$\Pi(s) = \sum_{i=1}^{n} P_0(s)P_i(-s)\left[P_0(-s)P_i(s)\right]_o. \tag{20}$$

The following property pertains to $\Pi(s)$.

Proposition 2 *Suppose Assumption 1 holds. Then,* $\Pi(s)$ *can be factorized as follows:*

$$\Pi(s) = As^r \bar{\Pi}_1(s)\bar{\Pi}_2(-s) \tag{21}$$

where A *is a real constant,* $r \geq 1$ *is an integer and* $\bar{\Pi}_1(s)$ *and* $\bar{\Pi}_2(s)$ *are uniquely determined monic Hurwitz polynomials.*

Remark 2 From the expression (20) of $\Pi(s)$ it is easily verified that $\bar{\Pi}_1(s)$ contains $P_0(s)$ as a factor.

Once the factorization (21) is performed, consider the functions

$$\Phi_e^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)} \tag{22}$$

defined for even r, and

$$\Phi_o^*(s) = \frac{\bar{\Pi}_1(s)}{\bar{\Pi}_2(s)}\ s^{\mathrm{sgn}A\ (-1)^{(r-1)/2}} \tag{23}$$

defined for odd r. The following result relates $\Phi_e^*(s)$ and $\Phi_o^*(s)$ to $\gamma^*(\omega)$.

Proposition 3 *Let* $\rho < \rho^*$ *and suppose Assumption 1 holds. Then,*

$$\gamma^*(\omega) = \begin{cases} \gamma_{\Phi_e^*}(\omega) & r \text{ even} \\ \gamma_{\Phi_o^*}(\omega) & r \text{ odd} \end{cases}. \tag{24}$$

It can be shown that $\Phi_e^*(s)$ and $\Phi_o^*(s)$ are indeed positive real. Thus, it is possible to perform a small perturbation of their coefficients in order to obtain strictly positive real rational functions. This leads to the main result of [18], which provides the solution of the $RSPR$ problem for the l_2 case.

Theorem 3 *Given the ellipsoidal set* $\mathcal{P}_\rho^2$*, let* $\rho < \rho^*$ *and suppose Assumption 1 holds. Then, for sufficiently small positive* ε *and* δ*, the rational function*

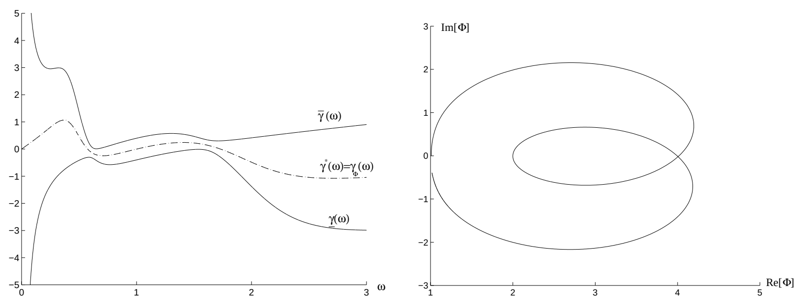
$$\Phi(s) = \begin{cases} \Phi_e^*(s)(1+\delta s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1} & \text{for even } r \\ \Phi_o^*(s)\left(\dfrac{s+\varepsilon}{s}\right)^{\mathrm{sgn}A\ (-1)^{(r-1)/2}} & \\ \cdot(1+\delta s)^{\partial\bar{\Pi}_2 - \partial\bar{\Pi}_1 - \mathrm{sgn}A\ (-1)^{(r-1)/2}} & \text{for odd } r \end{cases} \tag{25}$$

satisfies conditions 1 and 2 of Theorem 2, i.e. the filter

$$F(s) = \frac{P_0(s)}{\Phi(s)}$$

solves the robust SPR problem for $\mathcal{P}_\rho^2$.

Theorem 3 provides the solution of the $RSPR$ problem via the factorization of $\Pi(s)$ in Proposition 2. This allows for the determination of an upper bound on the degree of denominator $D_F(s)$ of $F(s)$ [18].



(a) (b)

Figure 2: (a): $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ (solid), $\gamma^*(\omega)$ (dotted), $\gamma_\Phi(\omega)$ (dashed); (b) Nyquist plot of $\overline{\Phi}_e^*(s)$.

Corollary 1 *Let the assumptions in Theorem 3 be fulfilled. Then,*

$$\begin{array}{ll} \partial D_F \leq m-2 & \text{for even } r \\ \partial D_F \leq m-1 & \text{for odd } r. \end{array} \tag{26}$$

Example 1 Consider the ellipsoidal set of polynomials

$$\mathcal{P}_\rho^2 = \left\{ P(s) = (s+1)^3 + q_1 s^2 + q_2 s \quad : \quad \|q\|_2 \leq \rho \right\}.$$

The vector functions in (6) are given by

$$R(\omega) = \frac{\left[-\omega^2(3-\omega^2) \quad \omega^2(1-3\omega^2)\right]'}{(1-3\omega^2)^2 + \omega^2(3-\omega^2)^2}$$
$$I(\omega) = \frac{\left[-\omega(1-3\omega^2) \quad -\omega^3(3-\omega^2)\right]'}{(1-3\omega^2)^2 + \omega^2(3-\omega^2)^2}.$$

Hence, the set $\mathcal{P}_\rho^2$ satisfies Assumption 1. According to Proposition 2, we have

$$\Pi(s) = -s^2(s+1)^4(s^2 - 0.78s + 3.54)(s^2 - 0.22s + 0.28)$$

and therefore $A = -1$, $r = 2$ and

$$\begin{array}{l} \overline{\Pi}_1(s) = (s+1)^4; \\ \overline{\Pi}_2(s) = (s^2 + 0.78s + 3.54)(s^2 + 0.22s + 0.28). \end{array}$$

Since r is even, according to (22) we have

$$\Phi_e^*(s) = \frac{(s+1)^4}{(s^2 + 0.78s + 3.54)(s^2 + 0.22s + 0.28)}.$$

In this case, $\Phi_e^*(s)$ is indeed strict positive real and therefore Theorem 3 leads to the rational filter

$$F(s) = \frac{(s^2 + 0.78s + 3.54)(s^2 + 0.22s + 0.28)}{s+1}.$$

solving the $RSPR$ problem for $\rho < \rho^* = \sqrt{7}$.
The function $\gamma_{\Phi_e^*}(\omega) = \gamma^*(\omega)$ is shown in Fig. 2(a) together with $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ for $\rho = 2.63$. Fig. 2(b) reports the Nyquist plot of $\Phi_e^*(s)$ making its SPR property clear.

5 l_∞ uncertainty case

In order to find a solution to the $RSPR$ problem for the class $\mathcal{P}_\rho^\infty$, according to the characterization in Theorem 2 we need to compute an SPR rational function $\Phi(s)$ such that $\gamma_\Phi(\omega)$ belongs to the set

$$\Gamma_\rho^\infty = \left\{ \gamma(\omega) \ : \ \|R(\omega) - \gamma(\omega) I(\omega)\|_1 < \frac{1}{\rho} \quad \forall \omega \geq 0 \right\}. \tag{27}$$

Notice that the inequality in (27) is nonsmooth as well as the function $\gamma^*(\omega)$ in (14), thus making the problem more difficult than in the l_2 case.
Nevertheless, a numerical filter design procedure which exploits Theorem 2 can be derived. Such procedure allows for the computation of a filter $F(s)$ of prescribed structure, e.g., a polynomial of degree n, solving the $RSPR$ problem with guaranteed robustness margin, i.e., for all $\rho < \rho_F$ with $\rho_F \leq \rho^*$. Moreover, the maximization of ρ_F can be carried out through a suitable tuning of the filter coefficients.
Although the characterization in Theorem 2 covers the case of the generic perturbation structure in (3), we restrict our discussion to the case in which the coefficients of $P(s) \in \mathcal{P}_\rho^\infty$ are perturbed independently, i.e., we consider the uncertain family

$$\mathcal{P}_\rho^\infty := \left\{ P(s) = P_0(s) + \sum_{i=1}^{n} q_i \hat{a}_{i-1} s^{i-1} : \|q\|_\infty \leq \rho \right\}$$

where $(\hat{a}_0, \ldots, \hat{a}_{n-1})' \in \mathbb{R}^n$ is a given vector. Clearly, this corresponds to choosing $P_i(s) = \hat{a}_{i-1} s^{i-1}$.
In this case, it is easily seen that the inequality in (27) can be rewritten as

$$\begin{aligned} & K_e(\omega) \left| \mathrm{Re}[P_0(j\omega)] + \gamma_\Phi(\omega) \mathrm{Im}[P_0(j\omega)] \right| \\ & + K_o(\omega) \left| \mathrm{Im}[P_0(j\omega)] - \gamma_\Phi(\omega) \mathrm{Re}[P_0(j\omega)] \right| < \frac{1}{\rho} \quad \forall \omega \geq 0, \end{aligned} \tag{28}$$

where

$$K_e(\omega) = \frac{1}{|P_0(j\omega)|^2} \sum_{i=0,\ i \text{ even}}^{n-1} \hat{a}_i \omega^i, \qquad K_o(\omega) = \frac{1}{|P_0(j\omega)|^2} \sum_{i=0,\ i \text{ odd}}^{n-1} \hat{a}_i \omega^i. \tag{29}$$

It has already been pointed out that for sufficiently small ρ, the filter $F(s) = P_0(s)$, i.e., $\Phi(s) = 1$, is indeed a solution to the $RSPR$ problem. Motivated by this observation, we introduce a class of polynomial filters of degree n which is obtained by performing a perturbation of the coefficients of $P_0(s)$. To this purpose, let

$$F(s;\theta) = P_0(s) + \sum_{i=0}^{n-1} \theta_i s^i \tag{30}$$

where $\theta = (\theta_0, \ldots, \theta_{n-1})' \in \mathbb{R}^n$ is a tunable parameter vector. Accordingly, define

$$\Phi(s;\theta) = \frac{P_0(s)}{F(s;\theta)}. \tag{31}$$

The related function $\gamma_\Phi(\omega;\theta) = \mathrm{Im}[\Phi(j\omega;\theta)]/\mathrm{Re}[\Phi(j\omega;\theta)]$ in (10) can be written as

$$\gamma_\Phi(\omega;\theta) = \frac{\mathrm{Im}[P_0(j\omega)]X_e(\omega;\theta) - \mathrm{Re}[P_0(j\omega)]X_o(\omega;\theta)}{1 + \mathrm{Re}[P_0(j\omega)]X_e(\omega;\theta) + \mathrm{Im}[P_0(j\omega)]X_o(\omega;\theta)} \tag{32}$$

where

$$\begin{aligned} X_e(\omega;\theta) &= \frac{1}{|P_0(j\omega)|^2} \sum_{i=0,\ i\ \text{even}}^{n-1} \theta_i (j\omega)^i \\ X_o(\omega;\theta) &= \frac{1}{j|P_0(j\omega)|^2} \sum_{i=0,\ i\ \text{odd}}^{n-1} \theta_i (j\omega)^i. \end{aligned} \tag{33}$$

Moreover, the set of $\theta \in \mathbb{R}^n$ such that $\Phi(s;\theta)$ is SPR is given by

$$\Theta_{\Phi+} = \left\{ \theta \in \mathbb{R}^n \ : \ \inf_{\omega \geq 0} [1 + \mathrm{Re}[P_0(j\omega)]X_e(\omega;\theta) + \mathrm{Im}[P_0(j\omega)]X_o(\omega;\theta)] > 0 \right\}. \tag{34}$$

In particular, the condition $\theta \in \Theta_{\Phi+}$ implies that $F(s;\theta)$ is Hurwitz.

Let us define

$$\begin{aligned} \rho_F(\theta)^{-1} \quad = \quad & \sup_{\omega \geq 0} \left(K_e(\omega) \left| \mathrm{Re}[P_0(j\omega)] + \gamma_\Phi(\omega;\theta)\mathrm{Im}[P_0(j\omega)] \right| \right. \\ & \left. + K_o(\omega) \left| \mathrm{Im}[P_0(j\omega)] - \gamma_\Phi(\omega;\theta)\mathrm{Re}[P_0(j\omega)] \right| \right) \end{aligned} \tag{35}$$

when $\gamma_\Phi(\omega;\theta)$ is as in (32)-(33). From (28), (35) and Theorem 2, we get that the filter $F(s;\theta)$ solves the $RSPR$ problem for $\mathcal{P}_\rho^\infty$ for all $\rho < \rho_F(\theta)$ provided that $\theta \in \Theta_{\Phi+}$.

The maximum perturbation norm $\rho_F(\theta)$ for which the filter $F(s;\theta)$ is guaranteed to provide a solution can be maximized with respect to the filter parameters through the solution of the optimization problem

$$\theta^* = \arg \sup_{\theta \in \Theta_{\Phi+}} \rho_F(\theta). \tag{36}$$

Such problem can be approached via a generic constrained search, initialized at $\theta = (0, \ldots, 0)'$, and involves, according to (35) a sweep along the ω axis

at each step.
It should be noted that $\rho_F(\theta^*)$ is in general a conservative lower bound of the parametric stability margin ρ^*. Moreover, its computation can be made difficult for the possible presence of local maxima, since the optimization problem (36) is non-convex in general. Nevertheless, a measure of the performance of the filter $F(s;\theta^*)$ can be obtained by comparing $\rho_F(\theta^*)$ with ρ^*, which can be computed explicitly [3]. Indeed, as remarked before, the condition $\rho < \rho^*$ is necessary and sufficient for the $RSPR$ problem to have a solution (not necessarily a polynomial one) [6].

Example 2 Consider the polynomial family $\mathcal{P}_\rho^\infty$ in (5) defined by

$$P_0(s) = s^6 + 2s^5 + 7.2s^4 + 7.2s^3 + 4.2s^2 + 2.2s + 0.4$$

and

$$(\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) = (0.2, 0.05, 0.05, 0.08, 0, 0).$$

The application of the proposed procedure yields

$$\theta^* = (-0.1212, 0, -0.0808, 0, -0.0404, 0)'$$

and $\rho(\theta^*) = 1.0590$. Hence, the filter

$$F_{\theta^*}(s) = s^6 + 2s^5 + 7.1596s^4 + 7.2s^3 + 4.1192s^2 + 2.2s + 0.2788$$

solves the $RSPR$ problem for all $\rho < \rho(\theta^*) = 1.0590$. It can be shown that $\rho(\theta^*)$ is indeed the actual l_∞ stability margin ρ^*.

5.1 l_∞ uncertainty: odd (even) perturbation case

A stronger synthesis result can be given under the additional condition that only the odd (even) coefficients of $P(s)$ are perturbed. Indeed, in this case, a filter $F(s)$ solving the $RSPR$ problem for all $\rho < \rho^*$ can be provided in closed form in the shape of a polynomial of degree n.
Let us introduce the following two subsets of $\mathcal{P}_\rho^\infty$

$$\mathcal{P}_\rho^{\infty,e} = \left\{ P(s) \in \mathcal{P}_\rho^\infty \quad : \quad \hat{a}_i = 0, \quad i \text{ even} \right\} \tag{37}$$

$$\mathcal{P}_\rho^{\infty,o} = \left\{ P(s) \in \mathcal{P}_\rho^\infty \quad : \quad \hat{a}_i = 0, \quad i \text{ odd} \right\} \tag{38}$$

which correspond to the odd and even perturbation case, respectively, and the two functions

$$\gamma_e(\omega) = \frac{\mathrm{Im}[P_0(j\omega)]}{\mathrm{Re}[P_0(j\omega)]} \quad ; \quad \gamma_o(\omega) = -\frac{\mathrm{Re}[P_0(j\omega)]}{\mathrm{Im}[P_0(j\omega)]}. \tag{39}$$

The following result is obtained.

Lemma 1 *Let $\rho < \rho^*$. Then, for all $\omega > 0$,*

$$\gamma^*(\omega) = \begin{cases} \gamma_e(\omega) & \text{for } \mathcal{P}_\rho^{\infty,e} \\ \gamma_o(\omega) & \text{for } \mathcal{P}_\rho^{\infty,o}. \end{cases} \tag{40}$$

Proof. It easily follows from (28), by observing that $K_e(\omega)$ (resp. $K_o(\omega)$) turns out to be zero. ♢

Indeed, there exist two rational functions $\Phi_e^*(s)$ and $\Phi_o^*(s)$ such that $\gamma_e(\omega) = \gamma_{\Phi_e^*}(\omega)$ and $\gamma_o(\omega) = \gamma_{\Phi_o^*}(\omega)$, respectively. Introduce the two polynomials

$$\begin{aligned} \Pi_e(s) &= -sP_0(s)[sP_0(-s)]_o \\ \Pi_o(s) &= P_0(s)[P_0(-s)]_o. \end{aligned}$$

By the properties of the real and imaginary parts of Hurwitz polynomials [3], it can be checked that $\Pi_o(s)$ and $\Pi_e(s)$ admit the following factorization

$$\Pi_e(s) = -s^2 P_0(s) \prod_{i=1}^{e_n} (s^2 + \omega_{e,i}^2) \tag{41}$$

$$\Pi_o(s) = -sP_0(s) \prod_{i=1}^{e_{n-1}} (s^2 + \omega_{o,i}^2) \tag{42}$$

where

$$e_n = \frac{n - n\text{mod } 2}{2}.$$

Note that the sets

$$\begin{aligned} \Omega_e^0 &= \{\omega_{e,1}, \ldots, \omega_{e,e_n}\} \\ \Omega_o^0 &= \left\{0, \omega_{o,1}, \ldots, \omega_{o,e_{n-1}}\right\} \end{aligned}$$

contain the frequencies at which $\mathrm{Re}[P_0(j\omega)] = 0$ and $\mathrm{Im}[P_0(j\omega)] = 0$, respectively. Introduce the two rational functions

$$\Phi_e^*(s) = \frac{P_0(s)}{\prod_{i=1}^{e_n} (s^2 + \omega_{e,i}^2)}$$

$$\Phi_o^*(s) = \frac{P_0(s)}{s \prod_{i=1}^{e_{n-1}} (s^2 + \omega_{o,i}^2)}.$$

The next result relates such functions to $\gamma_e(\omega)$ and $\gamma_o(\omega)$. The proof is similar to that of Lemma 6 in [18] and basically follows from expressions (41) and (42).

Theorem 4 *Let $\rho < \rho^*$. Then, $\gamma_e(\omega) = \gamma_{\Phi_e^*}(\omega)$ and $\gamma_o(\omega) = \gamma_{\Phi_o^*}(\omega)$. Moreover, $\Phi_e^*(s)$ and $\Phi_o^*(s)$ are positive real.*

Since $\Phi_e^*(s)$ and $\Phi_o^*(s)$ are shown to be positive real, suitable strictly positive real rational functions can be obtained via a small perturbation of their coefficients. Indeed, consider the two rational functions

$$\Phi_e(s) = \frac{P_0(s)}{\prod_{i=1}^{e_n}(s^2+2\zeta\omega_{e,i}s+\omega_{e,i}^2)(1+\delta s)^{n\mathrm{mod}\ 2}}$$

$$\Phi_o(s) = \frac{P_0(s)}{(s+\varepsilon)\prod_{i=1}^{e_{n-1}}(s^2+2\zeta\omega_{o,i}s+\omega_{o,i}^2)(1+\delta s)^{(n-1)\mathrm{mod}\ 2}}$$

and the corresponding two polynomial filters

$$\begin{aligned} F_e(s) &= \prod_{i=1}^{e_n}(s^2+2\zeta\omega_{e,i}s+\omega_{e,i}^2)(1+\delta s)^{n\mathrm{mod}\ 2} \\ F_o(s) &= (s+\varepsilon)\prod_{i=1}^{e_{n-1}}(s^2+2\zeta\omega_{o,i}s+\omega_{o,i}^2)(1+\delta s)^{(n-1)\mathrm{mod}\ 2} \end{aligned} \tag{43}$$

where ε, ζ, and δ are sufficiently small positive constants. We have the following result.

Theorem 5 *Let $\rho < \rho^*$. Then, the RSPR problem for the set $\mathcal{P}_\rho^{\infty,e}$ (resp. $\mathcal{P}_\rho^{\infty,o}$) is solved by the polynomial filter $F_e(s)$ (resp. $F_o(s)$) in (43).*

Proof (sketch): First, $\Phi_e(s)$ and $\Phi_o(s)$ can be proven to be strictly positive real. Moreover, the parameters ε, ζ and δ can be chosen such that $\gamma_{\Phi_e}(\omega)$ and $\gamma_{\Phi_o}(\omega)$ are arbitrarily close to $\gamma_e(\omega)$ and $\gamma_o(\omega)$, respectively. Then, the result follows by Lemma 1. $\diamondsuit$

6 l_1 uncertainty case

Let us assume $p = 1$. The set Γ_ρ^1 is given by

$$\Gamma_\rho^1 = \left\{\gamma(\omega)\ :\ \|R(\omega)-\gamma(\omega)I(\omega)\|_\infty < \frac{1}{\rho} \quad \forall\omega\geq 0\right\}. \tag{44}$$

It is easily checked that the inequality in (44) can be rewritten as

$$|R_i(\omega)-\gamma(\omega)I_i(\omega)| < \frac{1}{\rho} \quad \forall\omega\geq 0 \quad \forall i=1,\dots n. \tag{45}$$

As in the l_∞ case, let us assume that the coefficients of the uncertain polynomial are perturbed independently, i.e.,

$$\mathcal{P}_\rho^1 := \left\{P(s) = P_0(s) + \sum_{i=1}^{n} q_i\hat{a}_{i-1}s^{i-1} : \|q\|_1 \leq \rho\right\}. \tag{46}$$

By a straightforward manipulation, it turns out that $\gamma(\omega)$ belongs to Γ_ρ^1 if and only if

$$J_i(\gamma(\omega),\omega) < \frac{1}{\rho} \quad \forall \omega \geq 0, \qquad \forall i = 0,\ldots,n-1 \tag{47}$$

where

$$J_i(\gamma,\omega) = \begin{cases} \dfrac{|\hat{a}_i|\omega^i|\mathrm{Re}[P_0(j\omega)] + \gamma \mathrm{Im}[P_0(j\omega)]|}{|P_0(j\omega)|^2} & i \text{ even} \\ \dfrac{|\hat{a}_i|\omega^i|\mathrm{Im}[P_0(j\omega)] - \gamma \mathrm{Re}[P_0(j\omega)]|}{|P_0(j\omega)|^2} & i \text{ odd.} \end{cases} \tag{48}$$

The characterization in (47)-(48) can be exploited numerically the same way as in the l_∞ case in order to obtain a polynomial filter with guaranteed robustness margin. Let us define $F(s;\vartheta)$ as in (30) and introduce the functional

$$\rho_F(\theta)^{-1} = \sup_{\omega \geq 0} \max_{i=0,\ldots,n-1} J_i(\gamma_\Phi(\omega;\theta),\omega), \tag{49}$$

where $\gamma_\Phi(\omega;\theta)$ is as in (32).
Again, from Theorem 2, we get that the filter $F(s;\theta)$ solves the $RSPR$ problem for $\mathcal{P}_\rho^1$ for all $\rho < \rho_F(\theta)$ provided that $\theta \in \Theta_{\Phi+}$, and hence a suboptimal filter $F(s;\theta^*)$ solving the l_1 $RSPR$ problem for all $\rho < \rho_F(\theta^*) \leq \rho^*$ can be obtained through the non-convex optimization problem (36) with $\rho_F(\theta)$ as in (49).
Finally, it is worth to note that also in the l_1 case, a polynomial filter $F(s)$ of degree n solving the $RSPR$ problem for all $\rho < \rho^*$ can be obtained provided that only the even (odd) coefficients of the polynomial $P(s)$ are affected by uncertainty.
Introduce the two sets

$$\mathcal{P}_\rho^{1,e} = \left\{P(s) \in \mathcal{P}_\rho^1 \quad : \quad \hat{a}_i = 0, \quad i \text{ even}\right\} \tag{50}$$

$$\mathcal{P}_\rho^{1,o} = \left\{P(s) \in \mathcal{P}_\rho^1 \quad : \quad \hat{a}_i = 0, \quad i \text{ odd}\right\} \tag{51}$$

and let $\gamma_e(\omega)$ and $\gamma_o(\omega)$ be as in (39).
The following result parallels Lemma 1.

Lemma 2 *Let $\rho < \rho^*$. Then, for all $\omega > 0$,*

$$\gamma^*(\omega) = \begin{cases} \gamma_e(\omega) & \text{for } \mathcal{P}_\rho^{1,e} \\ \gamma_o(\omega) & \text{for } \mathcal{P}_\rho^{1,o}. \end{cases} \tag{52}$$

Proof. It directly follows from (44)-(48). ♢

It is then clear that a synthesis result as Theorem 5 can be derived.

Theorem 6 *Let $\rho < \rho^*$. Then, the RSPR problem for the set $\mathcal{P}_\rho^{1,e}$ (resp. $\mathcal{P}_\rho^{1,o}$) is solved by the polynomial filter $F_e(s)$ (resp. $F_o(s)$) in (43).*

7 Conclusion

The continuous-time robust SPR ($RSPR$) synthesis problem for l_p uncertain polynomials has been considered in this paper. It has been shown that a necessary and sufficient condition for the problem to have a solution is that all the polynomials in the uncertain family are Hurwitz. Under this assumption, a complete frequency domain characterization of the filters solving the $RSPR$ problem has been given. Such a characterization has been exploited in order to derive synthesis procedures for l_2, l_∞, and l_1 uncertainty, which provide filters with degree bounded by that of the uncertain polynomial. In the l_2 case, a closed form rational solution is devised. For l_∞ and l_1 uncertain polynomials, a numerical procedure is proposed to compute a polynomial filter with guaranteed robustness margin. In the last two cases, it has been shown that the sought filter can be computed in closed form provided that only the even (odd) coefficients of the uncertain polynomial are perturbed.

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