Synthesis of robust strictly positive real discrete-time systems with l_2 parametric perturbations

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Abstract— **This paper addresses the robust strict positive realness ($RSPR$) problem for families of discrete-time polynomials where the uncertainty is described via a l_2 ball in coefficient space. It is shown constructively that under the only assumption that all the polynomials of the family are Schur, the sought filter can be provided in closed form as a polynomial or rational function with an a-priori bounded degree. The proposed synthesis procedure is based on the solution of a simple polynomial factorization problem.**

Keywords— **Strict positive realness, adaptive schemes, passivity, robustness, discrete-time systems.**

I. Introduction

The problem of robust strict positive realness ($RSPR$) is of paramount importance in the framework of recursive schemes for identification and adaptive control of uncertain systems. In such schemes, providing robust local convergence relies on the design of a filter which ensures strict positive realness of a family of rational functions associated with plant uncertainty [1],[2]. More specifically, given a set $\mathcal{P}$ of polynomials in the z^{-1} variable, the $RSPR$ problem amounts to determining a polynomial or a rational filter $F(z^{-1})$ such that each rational function $P(z^{-1})/F(z^{-1}),\ P(z^{-1}) \in \mathcal{P}$ is strictly positive real. Several approaches have been proposed to deal with the $RSPR$ problem in both the continuous- and discrete-time case for different choices of the uncertainty set $\mathcal{P}$. In [3],[4],[5] a polyhedral set $\mathcal{P}$ in coefficient space is considered, while in [6] the set $\mathcal{P}$ is described in terms of root location regions. All the above results provide either sufficient or necessary and sufficient conditions for the existence of the $RSPR$ filter, and a few of them deal with filter synthesis. Notably, necessary and sufficient conditions providing the filter in closed form have not been given yet. In the recent paper [7] the case of $\mathcal{P}$ being an ellipsoid in coefficient space has been considered in the continuous-time framework. A constructive necessary and sufficient condition is given for the existence of a solution to the $RSPR$ problem, yielding a closed form expression for the sought filter.

In this paper, the discrete-time counterpart of the problem dealt with in [7] is considered. Exploiting technical properties which pertain to discrete-time polynomials and rational functions and which cannot be directly derived from the continuous-time case in [7], a necessary and sufficient condition for the existence of a solution to the discrete-time $RSPR$ problem with l_2 parametric uncertainty is derived. Moreover, a procedure for computing $F(z^{-1})$ in closed form is given. Such procedure is based on the solution of a polynomial factorization problem and yields a rational filter with bounded degree.

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The paper is organized as follows. Section 2 contains the problem formulation. The main result is introduced in Section 3 and the complete proof, which requires a few preliminary lemmas, is reported in Section 4. Some application examples are discussed in Section 5. Finally, Section 6 reports some concluding comments.

Notation.

$\mathbb{C}$:	the complex plane;
$z \in \mathbb{C}$:	complex number;
$\mathrm{Re}[z], \mathrm{Im}[z]$:	real and imaginary parts of z;
$P(z^{-1})$:	real polynomial in z^{-1};
$\mathbb{R}^n$:	real n-space;
$v = (v_1, \ldots, v_n)'$:	vector of $\mathbb{R}^n$ ($'$ denotes transpose);
$\|v\|_2$:	2-norm of v;
$\mathcal{S}$:	the set of Schur polynomials, i.e. the set of all polynomials in z^{-1} whose roots lie in $\|z\| < 1$;
$\mathrm{Res}[\Phi, z_0]$:	residue of function $\Phi(z^{-1})$ in $z = z_0$;
$u[x]$:	the unit step function;
a mod b:	remainder of a/b.

II. Problem formulation and preliminary results

First, we need to recall the definition of positive realness (PR) and strict positive realness (SPR) of a discrete-time rational transfer function.

Definition 1: A discrete-time rational transfer function $\Phi(z^{-1})$ is said to be positive real (PR) if
1. $\Phi(z^{-1})$ is analytic in $|z| > 1$;
2. $\mathrm{Re}[\Phi(z^{-1})] \geq 0 \quad \forall z \ : \ |z| > 1$.

Definition 2: A discrete-time rational transfer function $\Phi(z^{-1}) = \dfrac{N_\Phi(z^{-1})}{D_\Phi(z^{-1})}$ is said to be strictly positive real (SPR) if
1. $N_\Phi(z^{-1}), D_\Phi(z^{-1}) \in \mathcal{S}$;
2. $\mathrm{Re}\left[\Phi(e^{-j\omega})\right] > 0 \quad \forall \omega \in [0, 2\pi]$.

The following property pertains to PR discrete-time rational transfer functions [10].

Lemma 1: The discrete-time rational transfer function $\Phi(z^{-1})$ is PR if and only if
1. All poles z^* of $\Phi(z^{-1})$ with $|z^*| = 1$ are simple;
2. $\mathrm{Re}[\Phi(e^{-j\omega})] \geq 0$ for all $\omega \in [0, 2\pi]$ at which $\Phi(e^{-j\omega})$ exists and is finite;
3. If $z_0 = e^{j\omega_0}$ is a pole of $\Phi(z^{-1})$, then

$$e^{-j\omega_0}\mathrm{Res}[\Phi, z_0] > 0.$$

The following result relates PR and SPR [10].

Lemma 2: Let $\Phi^*(z^{-1})$ be PR. Then, for sufficiently small $\varepsilon > 0$,

$$\Phi(z^{-1}) = \Phi^*((1-\varepsilon)z^{-1})$$

is SPR.

Consider the set of polynomials of degree m

$$\mathcal{P}_\rho := \left\{ P(z^{-1}) = P_0(z^{-1}) + \sum_{i=1}^{n} q_i P_i(z^{-1}) : \|q\|_2 \leq \rho \right\}$$

where $P_0(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_m z^{-m},\ P_i(z^{-1})$, $i = 1, \ldots, n$ are given polynomials of degree m such that

$P_i(0) = 0$, $q = (q_1 \dots q_n)' \in \mathbb{R}^n$ is the parameter vector, and ρ is a positive scalar.
We can formulate the $RSPR$ problem as follows.

$RSPR$ problem. Given the set $\mathcal{P}_\rho$, determine, if it exists, a polynomial or a rational function $F(z^{-1})$ such that the rational function $P(z^{-1})/F(z^{-1})$ is SPR for all $P(z^{-1}) \in \mathcal{P}_\rho$.

Since the numerator of a strictly positive real rational rational function is Schur by definition, we enforce the following condition on $\mathcal{P}_\rho$.

Assumption 1: The nominal polynomial is Schur, i.e., $P_0(z^{-1}) \in \mathcal{S}$.
Let ρ^* denote the l_2 parametric stability margin of $\mathcal{P}_\rho$, i.e., the maximal ρ such that $\mathcal{P}_\rho$ contains all Schur polynomials

$$\rho^* = \sup_{\mathcal{P}_\rho \subset \mathcal{S}} \rho.$$

In [3] it was shown that $\rho < \rho^*$ (i.e. the family contains only Schur polynomials) is a necessary and sufficient condition for the existence of a solution to the $RSPR$ problem when the polynomial family is a polyhedron in coefficient space. In addition, a solution was provided in the form of a series expansion, thus yielding in general a polynomial filter of quite large and a-priori unknown degree.
In the present paper, we develop a procedure leading to direct closed-form synthesis of a filter $F(z^{-1})$, under the only requirement $\rho < \rho^*$. In particular, the filter $F(z^{-1})$ is computed in the shape of polynomial o rational function with degree bounded by a linear function of the degree m of $\mathcal{P}_\rho$.
In this respect, we note that some of the results which characterize the problem are straightforward extensions of the continuous-time case in [7], and therefore, they will not be worked out in deep detail. On the contrary, attention will be devoted to the aspects of the problem which are peculiar to the discrete-time case and indeed exploit technical features of discrete-time systems.
Moreover, due to space limitations, we will not consider all possible structures of $\mathcal{P}_\rho$, including non-generic ones, as it was done in [7], but we will restrict to the generic case only, in accordance with Assumption 2 in Section 3.
To proceed, let us introduce the following rational vector function

$$G(z^{-1}) = -\left[\frac{P_1(z^{-1})}{P_0(z^{-1})}, \dots, \frac{P_n(z^{-1})}{P_0(z^{-1})}\right]' \qquad (1)$$

and its real and imaginary parts evaluated along the unit circle

$$R(\omega) = \mathrm{Re}[G(e^{-j\omega})]; \quad I(\omega) = \mathrm{Im}[G(e^{-j\omega})].$$

It is easily verified that the $RSPR$ problem amounts to the computation of a SPR rational function $\Phi(z^{-1})$ such that

$$\mathrm{Re}\left[\Phi(e^{-j\omega})\left(1 - q'G(e^{-j\omega})\right)\right] > 0 \;\; \forall \omega \in [0, 2\pi] \;\; \forall q : \|q\|_2 \le \rho.$$

Once $\Phi(z^{-1})$ has been computed, $F(z^{-1})$ is readily obtained from the relation

$$F(z^{-1}) = \frac{P_0(z^{-1})}{\Phi(z^{-1})}.$$

Given a rational function $\Phi(z^{-1})$, define the following function of ω

$$\gamma_\Phi(\omega) = \frac{\mathrm{Im}[\Phi(e^{-j\omega})]}{\mathrm{Re}[\Phi(e^{-j\omega})]}.$$

The following result, which is a straightforward extension of Lemma 3 in [7], further characterizes the solution of the $RSPR$ problem.

Lemma 3: Any filter solving the $RSPR$ problem has the form

$$F(z^{-1}) = \frac{P_0(z^{-1})}{\Phi(z^{-1})}$$

where $\Phi(z^{-1})$ is a rational function such that
1. $\Phi(z^{-1})$ is SPR;
2. the inequality

$$\|R(\omega) - \gamma(\omega) I(\omega)\|_2 < \frac{1}{\rho} \quad \forall \omega \in [0, 2\pi] \qquad (2)$$

holds for $\gamma(\omega) = \gamma_\Phi(\omega)$.

III. Main result

The following technical assumption on the structure of $\mathcal{P}_\rho$ is enforced.

Assumption 2: The set $\mathcal{P}_\rho$ is such that $I(\omega) \neq 0$ for all $\omega \in (0, \pi) \cup (\pi, 2\pi)$.
It is easily verified that Assumption 2 indeed represents the generic case. The key observation is that each $\omega \notin \{0, \pi\}$ for which $I(\omega) = 0$ must be a common root of n polynomials in $e^{j\omega}$.
We are now ready to state the main result of the paper, which yields the solution to the $RSPR$ problem.

Theorem 1: Let $\rho < \rho^*$ and enforce Assumption 2. Consider the polynomial

$$\Pi(z^{-1}) = \sum_{i=1}^{n} P_0(z) P_i(z^{-1}) \left[P_0(z^{-1}) P_i(z) - P_0(z) P_i(z^{-1})\right]. \qquad (3)$$

Then,
1. $\Pi(z^{-1})$ admits the following factorization

$$\Pi(z^{-1}) = A z^{-k} (1 - z^{-1})^r (1 + z^{-1})^s \bar{\Pi}_1(z) \bar{\Pi}_2(z^{-1}) \qquad (4)$$

where A is a real constant, k, $r \ge 1$ and $s \ge 1$ are integers and $\bar{\Pi}_1(z^{-1}), \bar{\Pi}_2(z^{-1})$ are uniquely determined monic Schur polynomials.
2. For sufficiently small $\varepsilon > 0$, the rational function

$$\Phi(z^{-1}) = \Phi^*((1 - \varepsilon) z^{-1}) \qquad (5)$$

where

$$\Phi^*(z^{-1}) = \frac{\sigma(z^{-1})(-1)^{\tau_r}}{z^{-(\tau_r + \tau_s)}(1 - z^{-1})^{r_0 \sigma(1)}(1 + z^{-1})^{s_0 \sigma(-1)}} \frac{\bar{\Pi}_1(z^{-1})}{\bar{\Pi}_2(z^{-1})} \qquad (6)$$

being

$$r_0 = r \bmod 2, \quad s_0 = s \bmod 2,$$

$$\kappa = \frac{r - r_0}{2} + \frac{s - s_0}{2} + k,$$

$$\sigma(z^{-1}) = \frac{\text{sgn } A \quad (-1)^{(r-r_0)/2}}{z^{-\kappa}},$$
$$\tau_r = r_0 u[-\sigma(1)], \quad \tau_s = s_0 u[-\sigma(-1)]$$

is SPR and such that $\gamma_\Phi(\omega)$ satisfies (2), i.e., the filter

$$F(z^{-1}) = \frac{P_0(z^{-1})}{\Phi(z^{-1})} \tag{7}$$

solves the $RSPR$ problem.

Remark 1: Note that the expression of the filter $F(z^{-1})$ in (7) does not depend on ρ, i.e., (7) is the sought solution for all $\rho < \rho^*$.

Remark 2: By simple inspection of (5) and (6) it is easily verified that the degrees of the numerator and denominator of the filter $F(z^{-1})$ are bounded above by a linear function of the degree m of the nominal polynomial $P_0(z^{-1})$. In particular, it can be shown that the degree of the numerator is bounded by $3m + 3$.

IV. Proof of Theorem 1

We start with the consideration that inequality (2) enforces a constraint on $\gamma(\omega)$ of the form

$$\underline{\gamma}(\omega) < \gamma(\omega) < \overline{\gamma}(\omega) \quad \forall \omega \in [0, 2\pi] \tag{8}$$

where $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$ are explicitly computable functions. Therefore, any solution of the $RSPR$ problem is given by a SPR function $\Phi(z^{-1})$ such that $\gamma_\Phi(\omega)$ lies within the band (8). Let us introduce the function

$$\gamma^*(\omega) = \arg \min_\gamma \|R(\omega) - \gamma I(\omega)\|_2$$

which is at each ω the value of $\gamma(\omega)$ minimizing the left hand side of (2).

Obviously, $\gamma^*(\omega)$ is contained in the band defined by $\underline{\gamma}(\omega)$ and $\overline{\gamma}(\omega)$. We will proceed by first showing that $\Phi^*(z^{-1})$ constructed as in (6) is PR and such that $\gamma_{\Phi^*}(\omega) = \gamma^*(\omega)$, and then by proving that $\Phi^*(z^{-1})$ can be perturbed as in (5) into a SPR function $\Phi(z^{-1})$ such that $\gamma_\Phi(\omega)$ satisfies (2).

Let us recall a fundamental result characterizing the l_2 parametric stability margin of the class $\mathcal{P}_\rho$ [9].

Lemma 4: Suppose Assumption 2 holds and let

$$\rho_0 = \frac{1}{\|R(0)\|_2} \tag{9}$$

$$\rho_\pi = \frac{1}{\|R(\pi)\|_2} \tag{10}$$

$$\bar{\rho} = \inf_{\omega \in (0,\pi) \cup (\pi, 2\pi)} \frac{\|I(\omega)\|_2}{[\|I(\omega)\|_2^2 \|R(\omega)\|_2^2 - (R'(\omega)I(\omega))^2]^{1/2}}. \tag{11}$$

Then, the l_2 parametric stability margin of $\mathcal{P}_\rho$ is given by

$$\rho^* = \min\{\rho_0, \rho_\pi, \bar{\rho}\}.$$

Next, we need to characterize the set Γ of all bounded functions $\gamma(\omega)$ solving (2). The following result parallels Lemma 4 in [7].

Lemma 5: Let ρ^* be the l_2 parametric stability margin of $\mathcal{P}_\rho$ and suppose $\rho < \rho^*$. Then, the following statements hold.

1. Γ is the set of bounded continuous functions $\gamma(\omega)$ such that
$$\underline{\gamma}(\omega) < \gamma(\omega) < \overline{\gamma}(\omega) \quad \forall \omega \in [0, 2\pi]$$
where
$$\underline{\gamma}(\omega) = \min\left\{\gamma^*(\omega) \pm \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}\right\};$$
$$\overline{\gamma}(\omega) = \max\left\{\gamma^*(\omega) \pm \frac{\sqrt{\Delta(\omega)}}{\|I(\omega)\|_2^2}\right\}$$
being
$$\gamma^*(\omega) = \frac{R'(\omega)I(\omega)}{\|I(\omega)\|_2^2} \tag{12}$$
and
$$\Delta(\omega) = [R'(\omega)I(\omega)]^2 - \|I(\omega)\|_2^2 \left[\|R(\omega)\|_2^2 - \frac{1}{\rho^2}\right].$$
2. Γ is nonempty.

Proof: The proof is the same as that of Lemma 4 in [7] with the set $\bar{\Omega}_0$ replaced by $\{(0, \pi) \cup (\pi, 2\pi)\}$ and the set Ω_0 replaced by $\{0, \pi\}$. ∎

The following lemma states two fundamental properties of the polynomial $\Pi(z^{-1})$ in (3) and its relationship with $\gamma^*(\omega)$, which is of utmost importance for the development of the main result.

Lemma 6: Let Assumption 2 hold. Then, the polynomial $\Pi(z^{-1})$ is such that

1.
$$\Pi(e^{-j\omega}) = 2\left[P_0(e^{j\omega})P_0(e^{-j\omega})\right]^2 \cdot [I'(\omega)I(\omega) - jR'(\omega)I(\omega)];$$

2.
$$\begin{array}{ll} \text{Re}\left[\Pi(e^{-j\omega})\right] \geq 0 & \forall \omega \in [0, 2\pi] \\ \text{Re}\left[\Pi(e^{-j\omega})\right] > 0 & \forall \omega \in (0, \pi) \cup (\pi, 2\pi) \end{array}; \tag{13}$$

3.
$$\gamma^*(\omega) = \frac{\text{Im}\left[\Pi(e^{j\omega})\right]}{\text{Re}\left[\Pi(e^{j\omega})\right]}.$$

Proof: Let $\Pi(z^{-1})$ be rewritten as

$$\begin{aligned} \Pi(z^{-1}) &= P_0(z)\left[-P_0(z^{-1})G(z^{-1})\right]' \\ &\quad \cdot \left[-P_0(z^{-1})P_0(z)G(z) + P_0(z)P_0(z^{-1})G(z^{-1})\right] \\ &= \left[P_0(z)P_0(z^{-1})\right]^2 \\ &\quad \cdot \left[G'(z^{-1})G(z) - G'(z^{-1})G(z^{-1})\right]. \end{aligned}$$

We get

$$\begin{aligned} \Pi(e^{-j\omega}) &= 2\left[P_0(e^{j\omega})P_0(e^{-j\omega})\right]^2 \\ &\quad \cdot [I'(\omega)I(\omega) - jR'(\omega)I(\omega)]. \end{aligned} \tag{14}$$

This proves statement 1. Statement 2 follows directly from (14) and Assumption 2. Finally, from (14) and (12) we get that statement 3 holds. ∎

The first statement of Theorem 1 follows from (13) and the fact that $I(\omega) = 0$ only for $\omega \in \{0, \pi\}$ (Assumption 2).

Now consider $\Phi^*(z^{-1})$ as in (6), we have the following result.

Lemma 7: The function $\Phi^*(z^{-1})$ has the following properties
1. $\Phi^*(z^{-1})$ is PR;
2. $\gamma_{\Phi^*}(\omega) = \gamma^*(\omega) \quad \forall\omega \in (0,\pi)\cup(\pi,2\pi)$.

Proof: We will first show that if two polynomials $\Pi_1(z^{-1})$ and $\Pi_2(z^{-1})$ are such that

$$\Pi(z^{-1}) = \Pi_1(z)\Pi_2(z^{-1})$$

then, the rational function

$$\Phi(z^{-1}) = \frac{\Pi_1(z^{-1})}{\Pi_2(z^{-1})}$$

is such that

1.
$$\mathrm{Re}[\Phi(e^{-j\omega})] > 0 \ \ \forall\omega \in (0,\pi)\cup(\pi,2\pi);$$

2.
$$\gamma_\Phi(\omega) = \gamma^*(\omega).$$

Indeed, we have

$$\Phi(e^{-j\omega}) = \frac{\Pi_1(e^{-j\omega})}{\Pi_2(e^{-j\omega})} = \frac{\Pi_1(e^{-j\omega})\Pi_1(e^{j\omega})}{\Pi_2(e^{-j\omega})\Pi_1(e^{j\omega})} =$$

$$= \frac{\|\Pi_1(e^{-j\omega})\|^2}{\Pi(e^{-j\omega})} = \frac{\|\Pi_1(e^{-j\omega})\|^2}{\Pi^*(e^{j\omega})} = \frac{\|\Pi_1(e^{-j\omega})\|^2\Pi(e^{j\omega})}{\|\Pi(e^{-j\omega})\|^2}.$$

Hence, $\mathrm{Re}[\Phi(e^{-j\omega})] > 0 \ \ \forall\omega \in (0,\pi)\cup(\pi,2\pi)$ by Lemma 6, and moreover

$$\gamma_\Phi(\omega) = \frac{\mathrm{Im}[\Phi(e^{-j\omega})]}{\mathrm{Re}[\Phi(e^{-j\omega})]} = \frac{\mathrm{Im}[\Pi(e^{j\omega})]}{\mathrm{Re}[\Pi(e^{j\omega})]} = \gamma^*(\omega).$$

Now, it is easy to check that $\Pi(z^{-1})$ can be rewritten as

$$\begin{aligned}\Pi(z^{-1}) &= \mathrm{sgn}\ A\ (-1)^{(r-r_0)/2}\\ &\cdot[|A|^{1/2}(1-z)^{(r-r_0)/2}(1+z)^{(s-s_0)/2}]\bar{\Pi}_1(z)\\ &\cdot[|A|^{1/2}(1-z^{-1})^{(r-r_0)/2}(1+z^{-1})^{(s-s_0)/2}]\bar{\Pi}_2(z^{-1})\\ &\cdot(1-z^{-1})^{r_0}(1+z^{-1})^{s_0}z^{-\kappa}.\end{aligned}$$

By observing that

$$\begin{aligned}(1-z^{-1})^{r_0} &= (1-z^{-1})^{r_0u[\sigma(1)]}(1-z^{-1})^{r_0u[-\sigma(1)]}\\ &= (1-z^{-1})^{r_0u[\sigma(1)]}(-1)^{\tau_r}z^{-\tau_r}(1-z)^{r_0u[-\sigma(1)]}\end{aligned}$$

$$\begin{aligned}(1+z^{-1})^{s_0} &= (1+z^{-1})^{s_0u[\sigma(-1)]}(1+z^{-1})^{s_0u[-\sigma(-1)]}\\ &= (1+z^{-1})^{s_0u[\sigma(-1)]}z^{-\tau_s}(1+z)^{s_0u[-\sigma(-1)]}\end{aligned}$$

we get

$$\Pi(z^{-1}) = \Pi_1^*(z)\Pi_2^*(z^{-1})$$

and

$$\Phi^*(z^{-1}) = \frac{\Pi_1^*(z^{-1})}{\Pi_2^*(z^{-1})}$$

where

$$\begin{aligned}\Pi_1^*(z) &= \mathrm{sgn}\ A\ (-1)^{(r-r_0)/2}\cdot|A|^{1/2}(1-z)^{(r-r_0)/2}(1+z)^{(s-s_0)/2}\\ &\cdot(-1)^{\tau_r}(1-z)^{r_0u[-\sigma(1)]}\cdot(1+z)^{s_0u[-\sigma(-1)]}\bar{\Pi}_1(z)\end{aligned}$$

$$\begin{aligned}\Pi_2^*(z^{-1}) &= |A|^{1/2}(1-z^{-1})^{(r-r_0)/2}(1+z^{-1})^{(s-s_0)/2}\\ &\cdot(1-z^{-1})^{r_0u[\sigma(1)]}(1+z^{-1})^{s_0u[\sigma(-1)]}\cdot z^{-\kappa}z^{-\tau_r}z^{-\tau_s}\bar{\Pi}_2(z^{-1}).\end{aligned}$$

Hence, $\gamma_{\Phi^*}(\omega) = \gamma^*(\omega)$ and $\mathrm{Re}[\Phi^*(e^{-j\omega})] > 0$ for all $\omega \in (0,\pi)\cup(\pi,2\pi)$.

Since a rational function is PR if and only if its inverse is, all we have to check is that $\Phi^*(z^{-1})$ is indeed PR by proving that either $\Phi^*(z^{-1})$ or its inverse have positive residues in $z=1$ and negative residues in $z=-1$. For the sake of simplicity, we check this only for the case

$$r_0 = s_0 = 1, \quad \sigma(1) = 1, \quad \sigma(-1) = -1.$$

We get $\tau_r = 1$, $\tau_s = -1$, then

$$\Phi^*(z^{-1}) = \frac{\sigma(z^{-1})(1+z^{-1})}{z^{-1}(1-z^{-1})}\frac{\bar{\Pi}_1(z^{-1})}{\bar{\Pi}_2(z^{-1})}.$$

We have

$$\mathrm{Res}[\Phi^*, 1] = 2\sigma(1)\frac{\bar{\Pi}_1(1)}{\bar{\Pi}_2(1)} > 0$$

$$\mathrm{Res}[\Phi^{*-1}, -1] = \frac{2}{\sigma(-1)}\frac{\bar{\Pi}_2(-1)}{\bar{\Pi}_1(-1)} < 0.$$

The other cases are handled the same way. ■

To prove the second part of Theorem 1, taking Lemma 3 into account, it suffices to show that $\Phi(z^{-1})$ in (5) is SPR and that the inequality

$$\|R(\omega) - \gamma(\omega)I(\omega)\|_2 < \frac{1}{\rho} \tag{15}$$

holds for $\gamma(\omega) = \gamma_\Phi(\omega)$ for all $\omega \in [0,2\pi]$.

According to Lemma 7, $\Phi^*(z^{-1})$ is PR. Hence, $\Phi(z^{-1})$ turns out to be SPR by Lemma 2.

By Lemma 5, Lemma 6 and Lemma 7, (15) is satisfied for $\gamma(\omega) = \gamma_{\Phi^*}(\omega)$ for any $\rho < \rho^*$ and $\omega \in (0,\pi)\cup(\pi,2\pi)$. Moreover, since $\gamma_\Phi(\omega)$ depends continuously on ε, it turns out that the left hand term of inequality (15) for $\gamma(\omega) = \gamma_\Phi(\omega)$ is continuous with respect to ε. Hence, observing that $\gamma_\Phi(0) = 0$ and $\rho < \rho^*$, it follows that, for sufficiently small ε, (15) holds for $\gamma(\omega) = \gamma_\Phi(\omega)$ for all $\omega \in [0,2\pi]$. This completes the proof of the main result.

Remark 3: In Theorem 1, the function $\Phi(z^{-1})$ is obtained by performing a small perturbation $z^{-1} \to (1-\varepsilon)z^{-1}$ on $\Phi^*(z^{-1})$ in order to make it SPR. Indeed, it can be easily seen that a function $\Phi(z^{-1})$ satisfying the same properties can be found as well by applying such perturbation only to the factors with zeroes on the unit circle (see also the examples in the next section).

V. Examples

In this section we present some examples to illustrate the proposed results.

Example 1: Consider the set

$$\mathcal{P}_\rho = \left\{P(z^{-1}) = (1-0.5z^{-1})^2 + q_1z^{-1} + q_2z^{-2} \ \ ; \ \ \|q\|_2 \le \rho\right\}$$

It is easy to verify that the l_2 parametric stability margin is given by $\rho^* = 1/(4\sqrt{2})$. Moreover, from (1) we have

$$G(z^{-1}) = \left[\frac{-z^{-1}}{(1-0.5z^{-1})^2}, \frac{-z^{-2}}{(1-0.5z^{-1})^2}\right]'$$

and from (3)

$$\begin{aligned}\Pi(z^{-1}) &= (1-0.5z)^2 z^{-1}[(1-0.5z^{-1})^2 z - (1-0.5z)^2 z^{-1}] \\ &+ (1-0.5z)^2 z^{-2}[(1-0.5z^{-1})^2 z^2 - (1-0.5z)^2 z^{-2}] \\ &= \tfrac{7}{4}(1+z^{-1})(1-z^{-1})(1-0.5z)^2(1-\tfrac{4}{7}z^{-1}+\tfrac{4}{7}z^{-2}).\end{aligned}$$

It is easily shown that Assumption 2 holds and the factorization 4 yields

$$\begin{gathered}A = 7/4; \quad r=1; \quad s=1; \\ \bar{\Pi}_1(z^{-1}) = (1-0.5z^{-1})^2; \quad \bar{\Pi}_2(z^{-1}) = (1-\tfrac{4}{7}z^{-1}+\tfrac{4}{7}z^{-2}); \\ r_0 = 1, \quad s_0 = 1, \quad \kappa = 0, \quad \sigma(z^{-1}) = 1, \quad \tau_r = \tau_s = 1.\end{gathered}$$

Therefore, from Theorem 1, we have that for sufficiently small $\varepsilon > 0$, the function $\Phi(z^{-1}) = \Phi^*((1-\varepsilon)z^{-1})$ where

$$\Phi^*(z^{-1}) = \frac{(1-0.5z^{-1})^2}{(1+z^{-1})(1-z^{-1})(1-\frac{4}{7}z^{-1}+\frac{4}{7}z^{-2})}$$

yields a filter $F(z^{-1})$ solving the $RSPR$ problem for all $\rho < 1/(4\sqrt{2})$. By Remark 3, a solution having a simpler expression can be derived by perturbing only the factors of $\Phi^*(z^{-1})$ with zeroes on the unit circle, yielding

$$\begin{aligned}\Phi(z^{-1}) &= \frac{(1-0.5z^{-1})^2}{(1-\frac{4}{7}z^{-1}+\frac{4}{7}z^{-2})} \\ &\cdot \frac{1}{(1+(1-\varepsilon)z^{-1})(1-(1-\varepsilon)z^{-1})}.\end{aligned}$$

The corresponding filter $F(z^{-1})$ is given by

$$F(z^{-1}) = (1+(1-\varepsilon)z^{-1})(1-(1-\varepsilon)z^{-1})(1-\frac{4}{7}z^{-1}+\frac{4}{7}z^{-2}).$$

Note that in this particular case the filter $F(z^{-1})$ is a polynomial. The plot of $\gamma_\Phi(\omega)$ is depicted in Fig. 1(a) and the Nyquist plot of $\Phi(z^{-1})$ is depicted in Fig. 1(b).

Example 2: Let

$$\begin{aligned}\mathcal{P}_\rho = \{&P(z^{-1}) = (1-0.5z^{-1})^2 + q_1 z^{-1} + q_2(z^{-1}+z^{-2}) \\ ; \; &\|q\|_2 \le \rho\}\end{aligned}$$

we have

$$\rho^* = 1/4\sqrt{5}.$$

Computing and factorizing $\Pi(z^{-1})$ yields

$$\begin{aligned}\Pi(z^{-1}) &= (1-0.5z)^2(1+z^{-1})(1-z^{-1}) \\ &\cdot(z+1.4565)(1+0.0435z^{-1}+0.6866z^{-2})\end{aligned}$$

hence, by Theorem 1 and Remark 3,

$$\begin{aligned}\Phi(z^{-1}) &= \frac{(1-0.5z^{-1})^2(1+0.6866z^{-1})}{)(1+0.0435z^{-1}+0.6866z^{-2})} \\ &\cdot \frac{1}{(1-(1-\varepsilon)z^{-1})(1+(1-\varepsilon)z^{-1})}\end{aligned}$$

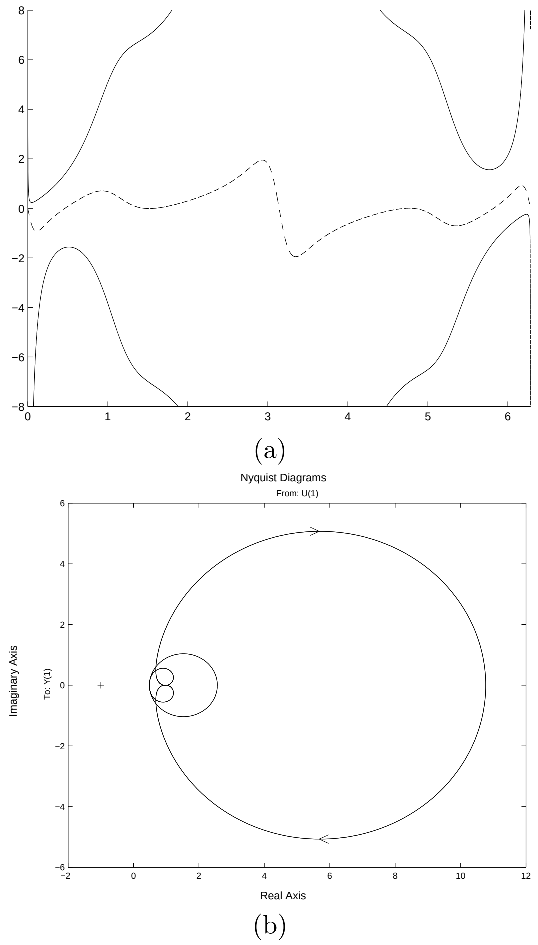

Fig. 1. Example 1. (a): Plot of $\gamma_\Phi(\omega)$ (dashed), $\overline{\gamma}(\omega)$, $\underline{\gamma}(\omega)$ (solid). (b): Nyquist plot of $\Phi(z^{-1})$

and $F(z^{-1})$ is then computed as

$$\begin{aligned}F(z^{-1}) &= \frac{(1+0.0435z^{-1}+0.6866z^{-2})}{1+0.6866z^{-1}} \\ &\cdot(1-(1-\varepsilon)z^{-1})(1+(1-\varepsilon)z^{-1})\end{aligned}$$

Note that in this case, the obtained filter is not a polynomial.

VI. Conclusion

The problem of designing discrete-time filters for robust strict positive realness of an uncertain family of polynomials ($RSPR$ problem) has been considered in the case of uncertainty being defined by an ellipsoid in coefficient space. It has been shown that a necessary and sufficient condition for a solution to exist is the stability of the polynomial family. The sought filter was provided in the shape of a polynomial or rational function with a-priori bounded degree. Moreover, it has been shown that the solution can be computed in closed form by solving a polynomial factorization problem. Finally, some application examples have been developed to illustrate the proposed results.

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