Linear fractional representations and $\mathcal{L}_2$-stability analysis of continuous piecewise affine systems

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***Abstract*—This paper addresses $\mathcal{L}_2$-stability analysis of discrete-time continuous piecewise affine systems described in input-output form by linear combinations of basis piecewise affine functions. The proposed approach exploits an equivalent representation of these systems as the feedback interconnection of a linear system and a diagonal static block with repeated scalar nonlinearity. This representation enables the use of analysis results for systems with repeated nonlinearities based on integral quadratic constraints. This leads to a sufficient condition for $\mathcal{L}_2$-stability that can be checked via the solution of a single linear matrix inequality, whose dimension grows linearly with the number of basis piecewise affine functions defining the system. Numerical examples corroborate the proposed approach by providing a comparison with an alternative approach based on the computation of piecewise polynomial storage functions.**

***Index Terms*—Continuous piecewise affine systems, linear fractional representations, $\mathcal{L}_2$-stability analysis, linear matrix inequalities.**

I. INTRODUCTION

Piecewise affine (PWA) systems are collections of linear/affine dynamics defined over a polyhedral partition of the system domain. Since PWA maps possess universal approximation properties [1], PWA systems represent an attractive model structure for system identification, combining the potential to approximate any nonlinear dynamics with the property of being locally linear [2]. While discontinuous PWA maps are more suited to model hybrid systems and systems with abrupt changes, approximation of continuous nonlinear dynamics calls for piecewise affine model structures embedding this property in their definition. One example is represented by hinging hyperplane (HH) functions, defined as the sum of a given number of hinge functions, each consisting in either the maximum or the minimum of two affine functions [3]. The class of HH functions is equivalent to the class of continuous PWA functions that can be expressed in Chua's canonical representation [4], [5], but there exist continuous PWA functions that cannot be expressed in this form. A universal representation of all continuous PWA functions can be obtained by linear combinations of Basis PWA (BPWA) functions [6]. An n-dimensional BPWA function is the maximum or minimum of $n+1$ affine functions. Hinge functions are therefore a particular case of BPWA functions. For a given level of accuracy, BPWA functions may approximate a given nonlinear map with fewer parameters than HH functions.

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This paper focuses on $\mathcal{L}_2$-stability analysis of basis piecewise affine autoregressive systems with exogenous inputs (BPWARX), i.e. discrete-time nonlinear regression models based on the use of BPWA functions. While identification of these models is tackled in [6], to the best of the authors' knowledge their stability analysis has never been addressed before. Using arguments from the dissipativity theory for nonlinear systems [7], a possible approach is to build a state-space realization of the BPWARX system, and then look for a suitable piecewise quadratic (PWQ) or piecewise polynomial (PWP) storage function ensuring bounded $\mathcal{L}_2$-gain of the resulting PWA model. Computation of storage functions can be tackled via convex optimization using linear matrix inequality (LMI) [8], [9] and sum-of-squares (SOS) [10] techniques. However, these approaches become computationally impracticable as the number of regions of the PWA model and, more importantly, the number of possible transitions between regions, grow. This is unfortunately likely to occur when PWA models are obtained as state-space realizations of BPWARX systems.

Approaches where the nonlinearities are pulled out of the system description, and characterized by means of sector bounded conditions and/or sector identities, are often applied in analysis problems to trade off computational complexity against conservatism, see, e.g., [11, Chap. 2], and more recently [12]. A similar approach is taken in this paper for $\mathcal{L}_2$-stability analysis of BPWARX systems. First, an equivalent linear fractional representation (LFR) of BPWARX systems is derived. It is shown that a given BPWARX system can be represented as the feedback interconnection of a linear system and a diagonal static block with repeated scalar nonlinearity. Using this representation, a sufficient condition for $\mathcal{L}_2$-stability of BPWARX systems is derived by exploiting analysis results for systems with repeated nonlinearities based on integral quadratic constraints (IQCs) [13]. The sufficient condition can be checked via a single LMI, whose dimension grows linearly with the number of BPWA functions defining the BPWARX system. The proposed approach is tested on numerical examples, and compared with the one in [10] based on the computation of PWQ/PWP storage functions. It is shown that, when the number of regions of the equivalent PWA model is small, the two approaches yield comparable results in terms of $\mathcal{L}_2$-stability margin (though the approach of this paper requires much smaller computation time). Conversely, when the number of regions grows, the approach in [10] fails due to out-of-memory issues caused by the huge size of the optimization problems involved, while the approach of this paper is still computationally affordable.

It is worthwhile to stress that, thanks to the universal representation properties of BPWA functions, the proposed approach is actually applicable to the whole class of discrete-time continuous PWA systems admitting an equivalent input-output representation [14]. The contribution of this paper extends our preliminary work in [15], where the focus was on $\mathcal{L}_2$-stability analysis of hinging hyperplane ARX (HHARX) systems. HHARX systems are indeed a subclass of the broader class of BPWARX systems.

The paper is structured as follows. The LFR of BPWARX systems is derived in Section II. Section III describes the proposed approach for $\mathcal{L}_2$-stability analysis of BPWARX systems. Numerical examples, including the comparison with the approach in [10], are reported in Section IV. Finally, conclusions are drawn in Section V.

Notation and preliminaries

The sets of real and nonnegative integer numbers are denoted by $\mathbb{R}$ and $\mathbb{Z}^{\geq 0}$, respectively. The standard notations for matrices and vectors are $A = \{a_{ij}\}$ and $x = \{x_i\}$, respectively. The $m \times n$ zero matrix is represented as $0_{m\times n}$, while the n-dimensional zero vector is 0_n. The $n \times n$ identity matrix is denoted I_n. The Kronecker product of matrices A and B is denoted by $A \otimes B$, while $\mathrm{diag}\{\ldots\}$ denotes block diagonal matrix composition.

The $\mathcal{L}_p$ space consists of all discrete-time signals u with finite $\mathcal{L}_p$-norm $\|u\|_{\mathcal{L}_p} := \left(\sum_{k=0}^{\infty} \|u(k)\|_p^p\right)^{\frac{1}{p}}$, where $\|\cdot\|_p$ denotes the p-norm of a vector. The $\mathcal{L}_{2,e}$ space is defined as $\mathcal{L}_{2,e} := \{u : u_{[0,\tau]} \in \mathcal{L}_2,\ \forall \tau \in \mathbb{Z}^{\geq 0}\}$, where $u_{[0,\tau]}$ is the truncated signal

$$u_{[0,\tau]}(k) := \begin{cases} u(k) & \text{if } 0 \leq k \leq \tau \\ 0 & \text{if } k > \tau. \end{cases} \quad (1)$$

The discrete-time unit impulse and unit step functions are denoted $\delta(k)$ and $\mathbf{1}(k)$, respectively. For a real discrete-time sequence $l(k)$, $k \in \mathbb{Z}^{\geq 0}$, $\hat{l}(z)$ denotes its zeta transform. The set of proper transfer matrices with all poles inside the open unit circle is denoted by $\mathcal{RH}_\infty$. If $\varepsilon > 0$, a scalar function $f : \mathbb{R} \to \mathbb{R}$ is said to belong to the sector $[0, \varepsilon]$ if $f(x)(\varepsilon x - f(x)) \geq 0$ for all $x \in \mathbb{R}$. Furthermore, let us introduce the function $\nu : \mathbb{R} \to \mathbb{R}$

$$\nu(x) := \max\{0, x\}, \quad (2)$$

which is the non-odd nonlinearity belonging to the sector $[0, 1]$ depicted in Fig. 1. The following property holds:

$$\begin{aligned} \max\{x_1, x_2\} &= x_1 + \max\{0, x_2 - x_1\} \\ &= x_1 + \nu(x_2 - x_1) \quad \forall x_1, x_2 \in \mathbb{R}. \end{aligned} \quad (3)$$

Finally, we define the vector function $\mathcal{N}_Q : \mathbb{R}^Q \to \mathbb{R}^Q$ as

$$\mathcal{N}_Q(x) := [\ \nu(x_1)\ \ldots\ \nu(x_Q)\ ]^T. \quad (4)$$

Notice that $\mathcal{N}_Q$ can be seen as a static multivariable input-output diagonal operator in which the scalar nonlinearity ν in Fig. 1 is repeated Q times.

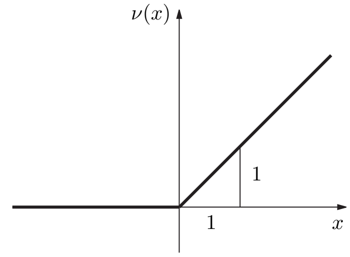

Fig. 1. Scalar nonlinearity repeated in the diagonal static block $\mathcal{N}_Q$.

II. Representations of BPWARX systems

We consider single-input single-output discrete-time continuous PWA systems described by BPWARX models [6]. In this section, the class of BPWARX systems is first introduced. Then, we develop a systematic procedure to cast any BPWARX system into LFR form.

A. BPWARX systems

Let $k \in \mathbb{Z}^{\geq 0}$ be the time index. A BPWARX system is described by the equation

$$\begin{aligned} y(k) &= \theta_0^T \phi(k) \\ &\quad + \textstyle\sum_{i=1}^{M} \sigma_i \max\{0, \theta_{i,1}^T \phi(k), \theta_{i,2}^T \phi(k), \ldots, \theta_{i,N_i}^T \phi(k)\}, \end{aligned} \quad (5)$$

where, for fixed orders n_a and n_b, the regression vector is

$$\begin{aligned} \phi(k) := [\ & y(k-1)\ \ldots\ y(k-n_a) \\ & u(k)\ u(k-1)\ \ldots\ u(k-n_b)\ v(k)\ ]^T, \end{aligned} \quad (6)$$

and $u(k) \in \mathbb{R}$ and $y(k) \in \mathbb{R}$ are the system input and output at time k, respectively. It is assumed that $u(k-\ell) = 0$ and $y(k-\ell) = 0$ if $k - \ell < 0$. In (6), $v(k) = \mathbf{1}(k)$ is a fictitious input introduced to manage the affine terms of the model. The parameters of model (5) are given by

$$\begin{aligned} & \theta_0 \in \mathbb{R}^{n_a+n_b+2} \\ & \theta_{i,j} \in \mathbb{R}^{n_a+n_b+2}, \quad j = 1, \ldots, N_i, \quad i = 1, \ldots, M \\ & \sigma_i \in \{-1, 1\},\ i = 1, \ldots, M, \end{aligned} \quad (7)$$

where $M \geq 1$ and $N_i \geq 1$, $i = 1, \ldots, M$.

Remark 1: As a special case, when $N_i = 1$ for all i, equation (5) describes the class of HHARX systems [3].

Remark 2: A given BPWARX system can have multiple representations of the form (5). For example, by (3), we have that $y(k) = \theta_0^T \phi(k) + \max\{0, \theta_{1,1}^T \phi(k)\}$ and $y(k) = (\theta_0 + \theta_{1,1})^T \phi(k) + \max\{0, -\theta_{1,1}^T \phi(k)\}$ represent the same system.

B. LFR of BPWARX systems

It is well-known that interconnections consisting of a finite number of linear time-invariant systems and static nonlinearities can be always represented in LFR form as in Fig. 2 (see, e.g., [16]), where the block $\mathcal{L}$ contains the linear dynamics and all nonlinearities are pulled out into the static block $\mathcal{N}$. In Fig. 2, $\tilde{u}(k)$ and $\tilde{y}(k)$ are the system input and output at time $k \in \mathbb{Z}^{\geq 0}$, respectively, while $\tilde{z}(k)$ and $\tilde{w}(k)$ are internal signals. In order to devise such a representation for BPWARX systems (5), we first need to introduce the following lemma.

Lemma 1: Given $x_1, x_2, \ldots, x_N \in \mathbb{R}$, let $w_0 = 0$, $x_{N+1} = 0$, and consider the recursion

$$\begin{aligned} z_j &= x_j - x_{j+1} + w_{j-1} \\ w_j &= \max\{0, z_j\} = \nu(z_j) \end{aligned} \tag{8}$$

for $j = 1, \ldots, N$. Then,

$$w_N = \max\{0, x_1, x_2, \ldots, x_N\}. \tag{9}$$

Proof. We first prove the statement

$$z_j = \max\{x_1, \ldots x_j\} - x_{j+1} \tag{10}$$

by induction. Clearly, (10) holds for $j = 1$. Assuming that it holds for the generic index j, we have

$$\begin{aligned} w_j &= \max\{0, \max\{x_1, \ldots x_j\} - x_{j+1}\} \\ z_{j+1} &= x_{j+1} - x_{j+2} + \max\{0, \max\{x_1, \ldots x_j\} - x_{j+1}\}. \end{aligned}$$

Then, by (3):

$$\begin{aligned} z_{j+1} &= \max\{x_{j+1}, \max\{x_1, \ldots x_j\}\} - x_{j+2} \\ &= \max\{x_1, \ldots x_{j+1}\} - x_{j+2}. \end{aligned}$$

It turns out that $z_N = \max\{x_1, \ldots x_N\}$, and hence (9). □

We now derive the dynamic linear part $\mathcal{L}$ and the static nonlinear part $\mathcal{N}$ describing the LFR form of the BPWARX system (5). For each $i = 1, \ldots, M$, we define recursively

$$\begin{aligned} \tilde{z}_{i,j}(k) &= (\theta_{i,j} - \theta_{i,j+1})^T \phi(k) + \tilde{w}_{i,j-1}(k) \\ \tilde{w}_{i,j}(k) &= \max\{0, \tilde{z}_{i,j}(k)\}, \quad j = 1, \ldots, N_i, \end{aligned} \tag{11}$$

with $\tilde{w}_{i,0}(k) = 0$ and $\theta_{i,N_i+1} = 0_{n_a+n_b+2}$. Then, by Lemma 1, we can rewrite (5) as:

$$y(k) = \theta_0^T \phi(k) + \textstyle\sum_{i=1}^{M} \sigma_i \tilde{w}_{i,N_i}(k). \tag{12}$$

By defining:

$$\begin{aligned} \tilde{z}_i(k) &:= \left[\begin{array}{ccc} \tilde{z}_{i,1}(k) & \ldots & \tilde{z}_{i,N_i}(k) \end{array} \right]^T, \\ \tilde{w}_i(k) &:= \left[\begin{array}{ccc} \tilde{w}_{i,1}(k) & \ldots & \tilde{w}_{i,N_i}(k) \end{array} \right]^T, \\ \Theta_i &:= \left[\begin{array}{cccc} (\theta_{i,1} - \theta_{i,2}) & \ldots & (\theta_{i,N_i-1} - \theta_{i,N_i}) & \theta_{i,N_i} \end{array} \right], \\ \Gamma_i &:= \left[\begin{array}{cc} 0_{1\times(N_i-1)} & 0 \\ I_{N_i-1} & 0_{(N_i-1)\times 1} \end{array} \right], \end{aligned} \tag{13}$$

we get from (11) that

$$\tilde{z}_i(k) = \Theta_i^T \phi(k) + \Gamma_i \tilde{w}_i(k) \tag{14}$$

and using (4):

$$\tilde{w}_i(k) = \mathcal{N}_{N_i}(\tilde{z}_i(k)). \tag{15}$$

Finally, by grouping:

$$\begin{aligned} \tilde{z}(k) &:= \left[\begin{array}{ccc} \tilde{z}_1(k)^T & \ldots & \tilde{z}_M(k)^T \end{array} \right]^T, \\ \tilde{w}(k) &:= \left[\begin{array}{ccc} \tilde{w}_1(k)^T & \ldots & \tilde{w}_M(k)^T \end{array} \right]^T, \\ \Theta &:= \left[\begin{array}{ccc} \Theta_1 & \ldots & \Theta_M \end{array} \right], \\ \Gamma &:= \operatorname{diag}\{\Gamma_1, \ldots, \Gamma_M\}, \\ \Sigma &:= \big[0_{1\times(N_1-1)} \; \sigma_1 \; 0_{1\times(N_2-1)} \; \sigma_2 \; \ldots \\ &\qquad\qquad \ldots \; 0_{1\times(N_M-1)} \; \sigma_M \; \big], \end{aligned} \tag{16}$$

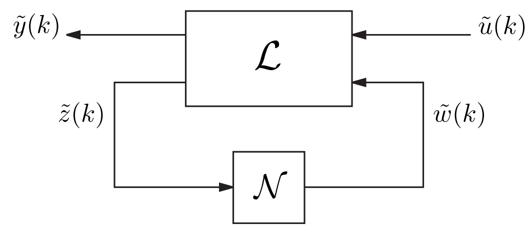

Fig. 2. LFR of an interconnected system consisting of a finite number of linear time-invariant systems and static nonlinear maps.

we have

$$\tilde{z}(k) = \Theta^T \phi(k) + \Gamma \tilde{w}(k), \tag{17}$$

and by (12),

$$y(k) = \theta_0^T \phi(k) + \Sigma \tilde{w}(k). \tag{18}$$

Moreover,

$$\tilde{w}(k) = \mathcal{N}_Q(\tilde{z}(k)), \tag{19}$$

where $Q = \sum_{i=1}^{M} N_i$. We have thus obtained that the BPWARX system (5) can be represented as in Fig. 2, where $\tilde{u}(k) := [\; u(k) \; v(k) \;]^T$ and $\tilde{y}(k) := y(k)$. The dynamic linear part $\mathcal{L}$ is defined by (17)-(18), while the static nonlinearity $\mathcal{N}$ is given by $\mathcal{N}_Q$ as per (19). The transfer functions that form the linear block $\mathcal{L}$ can be easily computed and are reported in equations (20) and (21) displayed in Table I, where θ_0 is decomposed as

$$\theta_0 = [\; -\alpha_1 \; \ldots \; -\alpha_{n_a} \; \beta_0 \; \beta_1 \; \ldots \; \beta_{n_b} \; \gamma_0 \;]^T, \tag{22}$$

and

$$d(z^{-1}) := 1 + \alpha_1 z^{-1} + \ldots + \alpha_{n_a} z^{-n_a}. \tag{23}$$

We recall that the input-output behaviors of the BPWARX system (5) are obtained by setting $v(k) = \mathbf{1}(k)$. In the following, to the purpose of deriving the sufficient condition for $\mathcal{L}_2$-stability, we will exploit the property that such v belongs to the $\mathcal{L}_{2,e}$ space.

III. $\mathcal{L}_2$-STABILITY ANALYSIS

The following standard notion of $\mathcal{L}_2$-stability is considered in this paper [17].

Definition 1: A discrete-time system with input $\tilde{u}$ and output $\tilde{y}$ is *finite-gain $\mathcal{L}_2$-stable* from $\tilde{u}$ to $\tilde{y}$ if there exists a nonnegative constant γ such that, for all $\tilde{u} \in \mathcal{L}_{2,e}$,

$$\|\tilde{y}_{[0,\tau]}\|_{\mathcal{L}_2} \leq \gamma \, \|\tilde{u}_{[0,\tau]}\|_{\mathcal{L}_2}, \quad \forall \tau \in \mathbb{Z}^{\geq 0}. \tag{24}$$

In the above definition, it is implicitly assumed that the output vanishes at $\tilde{u} = 0$. Otherwise, a bias term should be added to the right-hand side of (24).

The LFR derived in the previous section can be exploited in order to assess $\mathcal{L}_2$-stability for the BPWARX system (5). In particular, the following result holds.

Theorem 1: If the LFR (17)-(19) is finite-gain $\mathcal{L}_2$-stable from $\tilde{u} = [\; u \;\; v \;]^T$ to $\tilde{y}$, then the BPWARX system (5) is finite-gain $\mathcal{L}_2$-stable from u to y.

TABLE I

TRANSFER FUNCTIONS OF THE LINEAR PART $\mathcal{L}$ OF FIG. 2.

$$Y(z) = \frac{\beta_0 + \beta_1 z^{-1} + \ldots + \beta_{n_b} z^{-n_b}}{d(z^{-1})} U(z) + \frac{\gamma_0}{d(z^{-1})} V(z) + \frac{\Sigma}{d(z^{-1})} \tilde{W}(z) := L_{yu}(z)U(z) + L_{yv}(z)V(z) + L_{y\tilde{w}}(z)\tilde{W}(z) \quad (20)$$

$$\begin{aligned}\tilde{Z}(z) = {} & \Theta^T \begin{bmatrix} z^{-1}L_{yu}(z) \\ \vdots \\ z^{-n_a}L_{yu}(z) \\ 1 \\ z^{-1} \\ \vdots \\ z^{-n_b} \\ 0 \end{bmatrix} U(z) + \Theta^T \begin{bmatrix} z^{-1}L_{yv}(z) \\ \vdots \\ z^{-n_a}L_{yv}(z) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} V(z) + \left\{ \Theta^T \begin{bmatrix} z^{-1}L_{y\tilde{w}}(z) \\ \vdots \\ z^{-n_a}L_{y\tilde{w}}(z) \\ 0_{1\times Q} \\ 0_{1\times Q} \\ \vdots \\ 0_{1\times Q} \\ 0_{1\times Q} \end{bmatrix} + \Gamma \right\} \tilde{W}(z) \\ := {} & L_{\tilde{z}u}(z)U(z) + L_{\tilde{z}v}(z)V(z) + L_{\tilde{z}\tilde{w}}(z)\tilde{W}(z) \end{aligned} \quad (21)$$

Proof. All input-output behaviors of the BPWARX system (5) are obtained from the LFR (17)-(19) by setting $v(k) = \mathbf{1}(k)$. Since $\mathbf{1} \in \mathcal{L}_{2,e}$, the result holds. □

The $\mathcal{L}_2$-stability analysis will be carried out under the following assumption.

Assumption 1: All transfer functions forming the linear block in the LFR, i.e., $L_{y,u}$, $L_{y,v}$, $L_{y,\tilde{w}}$, $L_{\tilde{z},u}$, $L_{\tilde{z},v}$, $L_{\tilde{z},\tilde{w}}$ defined by (20)-(21), belong to $\mathcal{RH}_\infty$.

Remark 3: Assumption 1 boils down to the requirement that $d(z^{-1})$ in (23) is a Schur polynomial. Such an assumption may not be as restrictive as it seems. Indeed, as pointed out in Remark 2, the same system admits different equivalent representations, some of which may satisfy Assumption 1, while others may not. See [15] for an example in the special case of HHARX systems.

To proceed, let us consider the feedback interconnection depicted in Fig. 3, where $L_{\tilde{z},\tilde{w}}$ is the transfer function matrix from $\tilde{w}$ to $\tilde{z}$ of the derived LFR, defined as in (21). The following is a standard result in stability theory (see, e.g, [17]).

Theorem 2: Under Assumption 1, if the loop in Fig. 3 is finite-gain $\mathcal{L}_2$-stable from $\eta := [\,\eta_1^T\ \eta_2^T\,]^T$ to $\zeta := [\,\zeta_1^T\ \zeta_2^T\,]^T$, then the LFR in Fig. 2 is finite-gain $\mathcal{L}_2$-stable from $\tilde{u}$ to $\tilde{y}$.

To assess $\mathcal{L}_2$-stability of the loop in Fig. 3, which in turn implies $\mathcal{L}_2$-stability of the BPWARX system (5) by Theorems 1 and 2, the results in [13] can be used, since the block $\mathcal{N}_Q$ is composed of repeated scalar non-odd monotonically nondecreasing sector nonlinearities. Such results require the choice of a set of $r \geq 1$ positive scalar sequences $l_q(k)$ with finite $\mathcal{L}_1$-norm, $q = 0, \ldots, r-1$, where r is chosen arbitrarily. These sequences act as multipliers to reduce the conservatism of the stability condition. The following theorem specializes to the problem at hand the LMI stability criterion originally developed in [13] for the case of continuous-time systems.

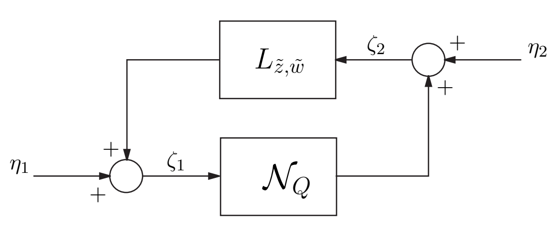

Fig. 3. Feedback configuration for stability analysis.

Theorem 3: Consider the loop in Fig. 3, with $L_{\tilde{z},\tilde{w}}$ defined in (21), and assume $L_{\tilde{z},\tilde{w}} \in \mathcal{RH}_\infty$. Let $l_q(k)$, $q = 0, \ldots, r-1$, be scalar positive sequences with finite $\mathcal{L}_1$-norm $\rho_q := \|l_q\|_{\mathcal{L}_1}$. Let (A, B, C, D) be a state space realization of the transfer function matrix $\Psi(z) \begin{bmatrix} L_{\tilde{z},\tilde{w}}(z) \\ I \end{bmatrix}$, where

$$\Psi(z) = \begin{bmatrix} I & -I \\ 0 & I \\ I & -I \\ 0 & I \\ \Upsilon(z) & -\Upsilon(z) \\ 0 & I \end{bmatrix}, \quad (25)$$

and $\Upsilon(z) := [\,\hat{l}_1(z)\ \ldots\ \hat{l}_r(z)\,]^T \otimes I$. If there exist real symmetric matrices $\Lambda_q := \{\lambda_{q_{ij}}\} \in \mathbb{R}^{Q\times Q}$, $q = 0, \ldots, r-1$, with positive entries, $Q \times Q$ real symmetric matrices $G^+ := \{g_{ij}^+\}$, $G^- := \{g_{ij}^-\}$, and P satisfying the following conditions:

$$\begin{aligned} & g_{ij}^+ \geq 0,\ g_{ij}^- \geq 0,\ \forall i,j = 1,\ldots,Q \\ & g_{ii}^- = 0,\ \forall i = 1,\ldots,Q \\ & g_{ij}^+ - g_{ij}^- \leq 0,\ \forall i,j = 1,\ldots,Q,\ i \neq j \\ & g_{ii}^+ \geq \sum_{\substack{j=1 \\ j\neq i}}^{Q} (g_{ij}^+ + g_{ij}^-) + \sum_{j=1}^{Q}\sum_{q=0}^{r-1} \rho_q \lambda_{q_{ij}},\ \forall i = 1,\ldots,Q \\ & \begin{bmatrix} A^TPA-P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} + [\,C\ D\,]^T W [\,C\ D\,] < 0,\quad P > 0, \end{aligned} \quad (26)$$

where

$$W := \begin{bmatrix} 0 & G^+ & 0 & 0 & 0 & 0 \\ G^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -G^- & 0 & 0 \\ 0 & 0 & -G^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -X^T \\ 0 & 0 & 0 & 0 & -X & 0 \end{bmatrix} \quad (27)$$

and $X := [\,\Lambda_0\ \ldots\ \Lambda_{r-1}\,]$, then the loop in Fig. 3 is finite-gain $\mathcal{L}_2$-stable from η to ζ. □

For fixed multipliers $l_q(k)$, $q = 0, \ldots, r-1$, the sufficient condition for $\mathcal{L}_2$-stability given in Theorem 3 can be checked via the solution of the LMI problem (26) in the unknowns G^+, G^-, P and Λ_q, $q = 0, \ldots, r-1$.

Remark 4: Combined with Theorems 1 and 2, Theorem 3 provides a sufficient condition for $\mathcal{L}_2$-stability of the BPWARX system (5) based on a generalization of the analysis in [13], which is developed for all repeated scalar monotonically nondecreasing nonlinearities belonging to a finite sector $[0, \varepsilon]$. Hence, a source of conservatism is introduced in the proposed approach by treating the static nonlinearity ν in Fig. 1 as a general sector nonlinearity when applying Theorem 3. Another source of conservatism comes from considering the fictitious input v in the construction of the LFR of Fig. 2. $\mathcal{L}_2$-stability of the LFR indeed implies to take into account all signals $v \in \mathcal{L}_{2,e}$, while the input-output behaviors of the BPWARX system (5) are obtained from the LFR with the particular choice $v(k) = \mathbf{1}(k)$. The conservatism of the proposed $\mathcal{L}_2$-stability condition can be reduced by exploiting the scalar positive sequences l_q of Theorem 3. Possible choices for these sequences are the discrete-time counterparts of the functions suggested in [13]. An example is shown in Section IV.

IV. Numerical examples

In this section, the $\mathcal{L}_2$-stability condition for BPWARX systems derived previously, is tested on three numerical examples. We also compare the performance of the proposed approach with that achieved using PWQ/PWP storage functions and a state-space representation of the BPWARX system, as in [10]. It is worthwhile to stress that, in the realization process, the number of regions of the equivalent state-space PWA model turns out to be typically much greater than the number of BPWA functions defining the BPWARX system. This has an impact on the computational requirements of the two approaches. While the $\mathcal{L}_2$-stability condition of this paper can be checked via the solution of a single LMI, whose dimension grows linearly with the number of BPWA functions defining the BPWARX system, the computation of PWQ/PWP storage functions requires to consider all possible transitions between regions.

Example 1: Let us consider the HHARX system (i.e., $N_i = 1\ \forall i = 1, \ldots, M$) characterized by $M = 4$, $n_a = 3$, $n_b = 1$, and parameters

$$\theta_0 = \begin{bmatrix} -0.8 & -0.5 & -0.3 & -0.1 & -0.7 & 0.1 \end{bmatrix}^T$$

$$\begin{bmatrix} \theta_{1,1} & \theta_{2,1} & \theta_{3,1} & \theta_{4,1} \end{bmatrix} = \begin{bmatrix} \alpha & 0.1 & 0.4 & 0.4 \\ 0.5 & 0.2 & 0.3 & 0.7 \\ 0.1 & 0.5 & 0.3 & 0.3 \\ -0.1 & 0.3 & -0.1 & 0.2 \\ 1 & -0.1 & 0.3 & -0.4 \\ 0.2 & -0.5 & 0.1 & 0.3 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix},$$

where $\alpha \geq 0$ is a scalar parameter. This system satisfies Assumption 1, since all the zeros of the polynomial

$$d(z^{-1}) = 1 + 0.8\, z^{-1} + 0.5\, z^{-2} + 0.3\, z^{-3} \tag{28}$$

belong to the open unit circle. We aim at computing an estimate of the set of positive α for which the system is $\mathcal{L}_2$-stable. More specifically, we perform a gridding over α with step equal to 0.01, and apply Theorem 3 in order to compute an interval contained in the $\mathcal{L}_2$-stability domain of the system. The conditions of Theorem 3 are tested using the SeDuMi [18] solver with CVX [19] as parser. The following choices are made for the multipliers $l_q(k)$:

a) No multipliers (e.g., $r = 1,\ l_0(k) = 0$);

b) $l_0(k) = \delta(k)$, $l_q(k) = (0.005q)^k,\ q = 1, \ldots, r-1$, for different values of $1 \leq r < 200$. Note that $l_q(k) \geq 0$ for all k, and $\rho_q = \|l_q\|_{\mathcal{L}_1} = (1 - 0.005q)^{-1}$.

With the choice *a*), we manage to prove $\mathcal{L}_2$-stability via Theorem 3 for $\alpha \in [0, 0.50]$, while the choice *b*) yields a stability interval $\alpha \in [0, 1.02]$ for $r = 2$, while no significant improvement is obtained for $r \geq 3$. In the latter case, the solution of the LMI problem of Theorem 3 involves 114 scalarized variables, 69 constraints, and takes about 0.6 s solver time and 1.48 s CVX time on a 3.6 GHz Intel i7-7700 processor.

Then, we compare the $\mathcal{L}_2$-stability region obtained via the proposed approach with that computed using the SOS relaxations presented in [10]. To this purpose, an equivalent PWA state-space representation with 16 regions is obtained using [20], and finiteness of the $\mathcal{L}_2$-gain is tested using PWP storage functions and polynomial relaxations. In order to obtain a comparable result, i.e., finite $\mathcal{L}_2$-gain for $\alpha \in [0, 1.12]$, a 4th order PWP storage function with 4th order polynomial relaxations has to be used (see Theorem 3.3 in [10] for details). The SOS problems are solved using Mosek (SeDuMi shows numerical problems on these tests) with YALMIP [21] as SOS problem parser, and results in 500640 scalarized variables and 315424 constraints. On the same hardware as above, solver time is 71.2 s (plus 2825 s YALMIP parsing time). The test fails for $\alpha > 1.12$. The use of PWQ storage functions always results in infeasible SOS problems.

Figure 4 shows the plot of the gain

$$\gamma(\tau) := \frac{\|y_{[0,\tau]}\|_{\mathcal{L}_2}}{\|u_{[0,\tau]}\|_{\mathcal{L}_2}}, \quad \tau \in \mathbb{Z}^{\geq 0}, \tag{29}$$

for different values of α and different input signals $u \in \mathcal{L}_{2,e}$. As expected, $\gamma(\tau)$ is bounded for $\alpha = 0.5$ and $\alpha = 1.01$, which are contained in the estimated $\mathcal{L}_2$-stability domain. Conversely, it turns out that $\alpha = 1.53$ does not belong to the actual stability domain, since $\gamma(\tau)$ is unbounded when the input is taken as $u(k) = \sin(\frac{\pi}{2}k)$. Notice that this cannot be concluded from the unit step response, as the corresponding $\gamma(\tau)$ remains bounded.

Example 2: We now consider the following HHARX system, characterized by $M = 8$, $n_a = 3$, $n_b = 1$, with

$$\theta_0 = \begin{bmatrix} -0.8 & -0.5 & -0.3 & -0.1 & -0.7 & 0.1 \end{bmatrix}^T$$

$$\begin{bmatrix} \theta_{1,1} & \ldots & \theta_{8,1} \end{bmatrix} = \begin{bmatrix} \alpha & 0.1 & 0.1 & 0.3 & 0.2 & 0.1 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.3 & 0.2 & 0.3 & 0.3 & 0.3 & 0.7 \\ 0.3 & 0.5 & 0.3 & 0.5 & 0.3 & 0.4 & 0.3 & 0.3 \\ -0.2 & 0.3 & -0.1 & 0.3 & -0.1 & 0.3 & -0.1 & 0.2 \\ 1 & -0.1 & 0.3 & -0.1 & 0.3 & -0.1 & 0.3 & -0.4 \\ 0.3 & -0.5 & 0.1 & -0.5 & 0.1 & -0.5 & 0.1 & 0.3 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 & \ldots & \sigma_8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

With the same choice of the multipliers as in the previous example, the proposed approach is able to assess an $\mathcal{L}_2$-stability interval $\alpha \in [0, 1.25]$. The corresponding LMI problem consists of 288 scalarized variables and 207 constraints. Solver time is 0.6 s and total CVX time is 1.61 s. An equivalent PWA state-space representation consists of 240 regions. This makes infeasible to apply the results in [10] with PWQ/PWP storage functions, as the formulation of the SOS problem results in out-of-memory issues.

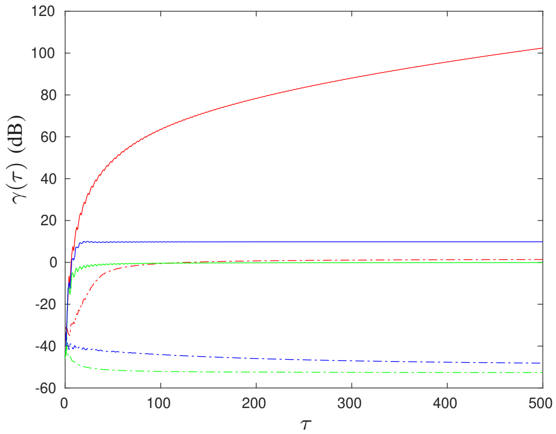

Fig. 4. Example 1: Plot of the gain $\gamma(\tau)$ for $\alpha = 0.5$ (green), $\alpha = 1.01$ (blue) and $\alpha = 1.53$ (red) using $u(k) = \mathbf{1}(k)$ (dash-dot) and $u(k) = \sin(\frac{\pi}{2}k)$ (solid).

Example 3: Let us consider the following BPWARX system, inspired by [6]:

$$\begin{aligned} y(k) = \alpha\,[&-0.5(y(k-1)+u(k-1)) \\ &+ \max\{0, y(k-1), u(k-1)\} \\ &- \max\{0, -y(k-1), -u(k-1)\}], \end{aligned}$$

where $\alpha > 0$. Using the results in [6], it can be shown that such a system cannot be represented by an HHARX model structure, regardless of the number of hinges used. On the other hand, the system is of the form (5) with $n_a = n_b = 1$, $M = 2$, $N_1 = N_2 = 2$, and

$$\theta_0 = \alpha \begin{bmatrix} -0.5 & 0 & -0.5 & 0 \end{bmatrix}^T$$

$$\begin{bmatrix} \theta_{1,1} & \theta_{1,2} & \theta_{2,1} & \theta_{2,2} \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

Using the proposed approach with the same choice of the multipliers as in Example 1, $\mathcal{L}_2$-stability can be assessed for $\alpha \in [0, 0.65]$.

V. Conclusions

A LFR of BPWARX systems was derived in this paper to enable the use of IQC-based analysis results for systems with repeated nonlinearities. Thanks to this transformation, a sufficient condition for $\mathcal{L}_2$- stability of BPWARX systems was presented. The proposed approach was compared via numerical examples with an alternative approach based on the computation of PWQ/PWP storage functions. It was shown that our approach is less computationally demanding than the latter, and succeeds in providing certifications of $\mathcal{L}_2$-stability even when the latter fails due to the intractable size of the optimization problems involved. When PWQ/PWP storage functions could be computed, the conservatism of the two approaches turned out to be comparable.

Future work aims at reducing the conservatism of the proposed approach (e.g., by incorporating in the stability conditions more information about the specific nonlinearities considered), as well as extending it to (subclasses of) discontinuous PWA systems.

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